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Research Article

Lebesgue points and summability of higher dimensional Fourier transforms

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ABSTRACT. The well known Lebesgue's theorem about the almost everywhere convergence of the one-dimensional Fejér means is generalized for five different, more general summability methods and for higher dimensional functions from the Wiener amalgam space $W(L_1, \ell_\infty)(\mathbb{R}^d)$.

Keywords: Fourier transforms, Fejér summability, θ-summability, Lebesgue points.

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1. INTRODUCTION

The classical theorem of Lebesgue [21] says that the Fejér means [11] of a one-dimensional integrable function f converge almost everywhere to f. More exactly,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T s_t f(x)\, dt = f(x)$$

at each Lebesgue point of f, thus almost everywhere, where $s_t f$ denotes the tth Dirichlet integral of the one-dimensional function f. In this survey paper, we generalize this theorem for higher dimensional functions and for more general summability methods. We give five different types of summability methods and five generalizations. All the five summability methods are investigated exhaustively in the literature.

A general method of summation, the so called θ-summation method, which is generated by a single function θ and which includes the well known Fejér, Riesz, Weierstrass, Abel, etc. summability methods, is studied intensively in the literature (see e.g. Butzer and Nessel [6], Trigub and Belinsky [2, 33, 34], Liflyand [23], Gát [12, 13, 14], Goginava [15, 16, 17], Simon [29], Persson, Tephnadze and Wall [27] and Weisz [35, 44]). Lebesgue points of multi-dimensional functions are investigated in Belinsky, Liflyand and Trigub [3, 1] and in Feichtinger and Weisz [9, 10].

For higher dimensional functions the θ-summability can be defined by

$$\sigma_T^{q,\theta} f(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \theta\left(\frac{\|t\|_q}{T}\right) \widehat{f}(t) e^{\imath x\cdot t}\, dt \qquad (x \in \mathbb{R}^d, T > 0)$$

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or by

$$\sigma_T^\theta f(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \prod_{j=1}^d \theta_j \left(\frac{|t_j|}{T_j} \right) \widehat{f}(t) e^{\imath x \cdot t} \, dt \qquad (x \in \mathbb{R}^d, T \in \mathbb{R}_+^d).$$

The special cases $q = 1, 2, \infty$ are investigated exhaustively in the literature. In particular, the case $q = 2$ in Stein and Weiss [31], Davis and Chang [8] and Grafakos [18], the case $q = 1$ in Berens [4, 5], Szili and Vértesi [32] and the case $q = \infty$ in Marcinkiewicz [24], Zhizhiashvili [45] and Weisz [35, 44]. The second type of summation was considered e.g. in Zygmund [46], Gát [12] and Weisz [35, 44].

In this paper, we introduce a different concept of Lebesgue points and a different Hardy-Littlewood maximal function for each summability just mentioned. We generalize Lebesgue's theorem for these new Lebesgue points and for the different summability methods and for higher dimensional functions from the Wiener amalgam space $W(L_1, \ell_\infty)(\mathbb{R}^d) \supset L_1(\mathbb{R}^d)$. All the results of this paper hold e.g. for the Weierstrass, Abel, Picard, Bessel, Fejér, de La Vallée-Poussin, Rogosinski and Riesz summations. This paper was the base of my talk given at the *3rd International Conference: Constructive Mathematical Analysis, July, 2025 in Konya (Türkiye).*

2. THE ONE-DIMENSIONAL θ-SUMMABILITY AND LEBESGUE'S THEOREM

For a fixed $d \in \mathbb{N}$, $d \geq 2$, we equip the space $L_p(\mathbb{R}^d)$ with the norm

$$\|f\|_p := \begin{cases} \left(\int_{\mathbb{R}^d} |f|^p \, d\lambda \right)^{1/p}, & 0 < p < \infty; \\ \sup_{\mathbb{R}^d} |f|, & p = \infty. \end{cases}$$

The Fourier transform of a one-dimensional function $f \in L_1(\mathbb{R})$ is given by

$$\widehat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-\imath x t} \, dt \qquad (x \in \mathbb{R}),$$

where $\imath = \sqrt{-1}$. If $f \in L_p(\mathbb{R})$ for some $1 \leq p \leq 2$ and $f \in L_1(\mathbb{R})$, then the Fourier inversion formula holds:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(t) e^{\imath x t} \, dt \qquad (x \in \mathbb{R}). \tag{2.1}$$

The integrability condition of $\widehat{f}$ is a very strong condition. Without this condition, we may consider the Dirichlet integral $s_T f$:

$$s_T f(x) := \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \widehat{f}(t) e^{\imath x t} \, dt,$$

which is well defined. It is known that for $f \in L_p(\mathbb{R})$, $1 < p < \infty$,

$$\lim_{T \to \infty} s_T f = f \qquad \text{in the } L_p(\mathbb{R})\text{-norm and a.e.}$$

The norm convergence is due to Riesz [28] and the almost everywhere convergence is the famous theorem due to Carleson and Hunt (see Carleson [7] and Hunt [19] or recently Grafakos [18]).

This convergence does not hold for $p = 1$. However, using a summability method, we can generalize these results. The most known summability method is the Fejér method. The Fejér means are defined by

$$\sigma_T f(x) := \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \left(1 - \frac{|t|}{T} \right) \widehat{f}(t) e^{\imath x t} \, dt = \frac{1}{T} \int_0^T s_t f(x) \, dt.$$

We generalize this summation and introduce a very general summability method, the so called θ-summation defined by a function $\theta : \mathbb{R}_+ \to \mathbb{R}$ satisfying $\theta(0) = 1$. This summation contains all well known summability methods, such as the well known Weierstrass, Abel, Picard, Bessel, Fejér, de La Vallée-Poussin, Rogosinski and Riesz summations. Similarly to the Fejér means, in the Fourier inversion formula (2.1), we multiply the integrand by a suitable function θ . More precisely, let

$$\sigma_T^\theta f(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \theta\left(\frac{|t|}{T}\right) \widehat{f}(t) e^{\imath x t}\, dt. \tag{2.2}$$

It is easy to see that we get back the Fejér means if $\theta(t) = \max((1-|t|), 0)$. This definition can easily be extended to all $f \in W(L_1, \ell_\infty)(\mathbb{R})$.

It is know that the Fejér means converge to f as $T \to \infty$ when $f \in L_1(\mathbb{R})$. However, we can characterize the set of convergence as follows. The Hardy-Littlewood maximal function is defined by

$$Mf(x) := \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x-t)|\, dt$$

and the following result holds (see e.g. Stein [30]):

Theorem 2.1. *If* $f \in L_1(\mathbb{R})$, *then*

$$\sup_{\rho>0} \rho \lambda(Mf > \rho) \le C\|f\|_1.$$

Moreover, if $1 < p \le \infty$ *and* $f \in L_p(\mathbb{R})$, *then*

$$\|Mf\|_p \le C_p \|f\|_p .$$

The weak type inequality of the theorem implies the next result due to Lebesgue.

Corollary 2.1. *If* $f \in L_1(\mathbb{R})$, *then*

$$\lim_{h\to 0} \frac{1}{2h} \int_{-h}^{h} f(x-t)\, dt = f(x) \qquad \textit{a.e. } x \in \mathbb{R}.$$

This convergence is equivalent to the next two convergences:

$$\lim_{h\to 0} \frac{1}{2h} \int_{-h}^{h} f(x-t) - f(x)\, dt = 0 \qquad \text{a.e.}$$

and

$$\lim_{h\to 0} \frac{1}{2h} \left| \int_{-h}^{h} f(x-t) - f(x)\, dt \right| = 0 \qquad \text{a.e.}\,.$$

If we can go with the absolute value inside the integral, then we say that x is a Lebesgue point of f. More exactly, a point $x \in \mathbb{R}$ is called a Lebesgue point of $f \in L_1(\mathbb{R})$ if

$$\lim_{h\to 0} \frac{1}{2h} \int_{-h}^{h} |f(x-t) - f(x)|\, dt = 0.$$

We can see, that the Lebesgue points are depending only on f and if f is continuous at x, then x is a Lebesgue point of f. All our generalizations will have these properties.

Theorem 2.2. *Almost every point* $x \in \mathbb{R}$ *is a Lebesgue point of* $f \in L_1(\mathbb{R})$.

The next well known theorem was proved by Lebesgue [21] in 1905 (see also [11]).

Theorem 2.3. *For all Lebesgue points of $f \in L_1(\mathbb{R})$,*

$$\lim_{T\to\infty} \sigma_T f(x) = f(x).$$

In what follows, we will generalize this theorem for multi-dimensional functions and for the general θ-summability method.

3. THE MULTI-DIMENSIONAL θ-SUMMABILITY

Let us turn to the higher dimensional functions. For $t, x \in \mathbb{R}^d$, let

$$t \cdot x := \sum_{k=1}^{d} t_k x_k, \qquad \|x\|_q := \begin{cases} \left(\sum_{k=1}^{d} |x_k|^q\right)^{1/q}, & 0 < q < \infty; \\ \sup_{i=1,\ldots,d} |x_i|, & q = \infty. \end{cases}$$

The Fourier transform of a higher dimensional function $f \in L_1(\mathbb{R}^d)$ is given by

$$\widehat{f}(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(t) e^{-\imath x \cdot t}\, dt \qquad (x \in \mathbb{R}^d).$$

The inversion formula (2.1) holds in this case, too.

The first question is, how can we generalize the definition (2.2) for higher dimensional functions? There are some generalizations, which are investigated exhaustively in the literature. In the first natural generalization, instead of $|t|$ we write the q-norm $\|t\|_q$ of the d-dimensional vector $t = (t_1, \ldots, t_d)$ and the function θ remains a one-dimensional function. These means are called ℓ_q-θ-means. More exactly, suppose that $\theta : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\theta \in C_0(\mathbb{R}_+), \qquad \theta(\|\cdot\|_q) \in L_1(\mathbb{R}^d), \qquad \theta(0) = 1. \tag{3.3}$$

Here $C_0(\mathbb{R}_+)$ denotes the set of continuous functions vanishing at infinity. The Tth ℓ_q-θ-mean of the function $f \in L_p(\mathbb{R}^d)$ $(1 \leq p \leq 2)$ is defined by

$$\sigma_T^{q,\theta} f(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \theta\left(\frac{\|t\|_q}{T}\right) \widehat{f}(t) e^{\imath x \cdot t}\, dt \qquad (x \in \mathbb{R}^d, T > 0).$$

Note that the integral is well defined and T is a positive real number. Usually three subcases are investigated in the literature: the ℓ_q-θ-means are called triangular if $q = 1$, circular if $q = 2$ and cubic if $q = \infty$ (see Figure 1). The cubic summability (when $q = \infty$) is also called Marcinkiewicz summability.

In the other natural generalization, instead of the function θ, we use d one-dimensional functions $\theta_1, \ldots, \theta_d$. The Tth rectangular θ-mean of the function $f \in L_p(\mathbb{R}^d)$ $(1 \leq p \leq 2)$ is given by

$$\sigma_T^{\theta} f(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \prod_{j=1}^{d} \theta_j\left(\frac{|t_j|}{T_j}\right) \widehat{f}(t) e^{\imath x \cdot t}\, dt \qquad (x \in \mathbb{R}^d, T \in \mathbb{R}_+^d).$$

In this case, we will always assume that

$$\theta = \theta_1 \otimes \cdots \otimes \theta_d, \qquad \theta_j \in L_1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+) \qquad \text{and} \qquad \theta_j(0) = 1 \tag{3.4}$$

for all $j = 1, \ldots, d$. In this definition, $T = (T_1, \ldots, T_d) \in \mathbb{R}_+^d$ is a vector. Two subcases of this summability will be investigated, the restricted (when $T \in \mathbb{R}_+^d$ is in a cone) and the unrestricted (when $T \in \mathbb{R}_+^d$) summability. For each of the five generalizations, we need a new concept of Lebesgue points. The proofs are strongly different for different cases.

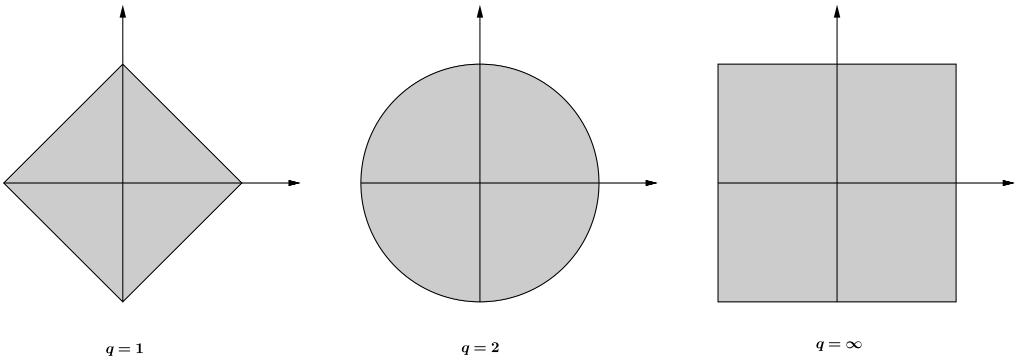

FIGURE 1. Regions of the ℓ_q-partial sums for $d = 2$.

4. NORM CONVERGENCE OF THE HIGHER DIMENSIONAL SUMMABILITY MEANS

In this section, we deal with the norm convergence of the ℓ_q-summability means and the rectangular means. First, we introduce the next function space. A Banach space B consisting of Lebesgue measurable functions on $\mathbb{R}^d$ is called a homogeneous Banach space if

(a) for all $f \in B$ and $x \in \mathbb{R}^d$, $T_x f \in B$ and $\|T_x f\|_B \sim \|f\|_B$,
(b) the function $x \mapsto T_x f$ from $\mathbb{R}^d$ to B is continuous for all $f \in B$,
(c) the functions in B are uniformly locally integrable, i.e., for every compact set $K \subset \mathbb{R}^d$ there exists a constant C_K such that

$$\int_K |f|\, d\lambda \leq C_K \|f\|_B \qquad (f \in B).$$

Note that if B is a homogeneous Banach space on $\mathbb{R}$, then

$$B \subset W(L_1, \ell_\infty)(\mathbb{R})$$

(see Katznelson [20]). It is easy to see that the spaces $C_0(\mathbb{R}^d)$, $L_p(\mathbb{R}^d)$, $W(L_p, \ell_q)(\mathbb{R}^d)$, $W(L_p, c_0)(\mathbb{R}^d)$, $W(C, \ell_q)(\mathbb{R}^d)$, $W_I(L_p, c_0)(\mathbb{R}^d)$ $(1 \leq p, q < \infty)$ and Lorentz spaces $L_{p,q}(\mathbb{R}^d)$ $(1 < p < \infty, 1 \leq q < \infty)$ are all homogeneous Banach spaces. Note that the definition of Wiener amalgam spaces can be found in Section 5. Moreover, the space $C_u(\mathbb{R}^d)$ of uniformly continuous bounded functions endowed with the supremum norm is also a homogeneous Banach space.

4.1. Norm convergence of the ℓ_q-summability means.

For an integrable function f,

$$\sigma_T^{q,\theta} f(x) = \int_{\mathbb{R}^d} f(x-t) K_T^{q,\theta}(t)\, dt = f * K_T^{q,\theta}(x) \qquad (x \in \mathbb{R}^d, T > 0),$$

where the Tth ℓ_q-θ-kernel is given by

$$K_T^{q,\theta}(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \theta\left(\frac{\|t\|_q}{T}\right) e^{\imath x \cdot t}\, dt = \frac{1}{(2\pi)^{d/2}} T^d \widehat{\theta_0}(Tx).$$

This definition can easily be extended to all $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$.

For $q=2$, we suppose in addition to (3.3) that

$$\widehat{\theta_0} \in L_1(\mathbb{R}^d), \tag{4.5}$$

where $\theta_0(x) := \theta(\|x\|_q)$. Since this condition is hard to check for $q=1$ and $q=\infty$, we use another concept in this case. Namely, we suppose that θ is continuous on $\mathbb{R}_+$, the support of θ is $[0,c]$ for some $0<c\leq\infty$ and θ is differentiable on $(0,c)$. Suppose further that

$$\theta(0)=1, \qquad \int_0^\infty \max\{t,1\}^d |\theta'(t)|\,dt<\infty, \qquad \lim_{t\to\infty} t^d\theta(t)=0 \tag{4.6}$$

and assume that

$$\left|\int_0^\infty \theta'(t)\,t\,\mathrm{soc}\,(tu)\,dt\right| \leq Cu^{-\alpha} \tag{4.7}$$

for all $u>0$ and for some $0<\alpha<\infty$, where

$$\mathrm{soc}\,t := \begin{cases} \cos t, & \text{if } d \text{ is even;} \\ \sin t, & \text{if } d \text{ is odd.} \end{cases}$$

We can show that (4.6) implies that $\theta(\|\cdot\|_q)\in L_1(\mathbb{R}^d)$ for $q=1$ and $q=\infty$.

The next two theorems were proved in Berens, Li and Xu [4], Oswald [26] and Weisz [38, 36, 37]. Moreover, Li and Xu [22] investigated these theorem for Jacobi polynomials.

Theorem 4.4. *Under the conditions* (3.3) *and* (4.5) *with $q=2$ or* (4.6) *and* (4.7) *with $q=1,\infty$, we have*

$$\int_{\mathbb{R}^d} \left|K_T^{q,\theta}(x)\right| dx \leq C \qquad (T>0).$$

Theorem 4.5. *Assume that B is a homogeneous Banach space on $\mathbb{R}^d$. Under the conditions of Theorem 4.4,*

$$\left\|\sigma_T^{q,\theta} f\right\|_B \leq C\,\|f\|_B \qquad (T>0)$$

and

$$\lim_{n\to\infty} \sigma_T^{q,\theta} f = f \qquad \text{in the } B\text{-norm for all } f\in B.$$

Since $C_u(\mathbb{R}^d)$ is a homogeneous Banach space, we obtain:

Corollary 4.2. *If f is a uniformly continuous and bounded function, then, under the conditions of Theorem 4.4,*

$$\lim_{T\to\infty} \sigma_T^{q,\theta} f = f \qquad \text{uniformly.}$$

4.2. **Norm convergence of the rectangular summability means.** For an integrable function f, the rectangular means can be written in the form

$$\sigma_T^\theta f(x) = \int_{\mathbb{R}^d} f(x-t) K_T^\theta(t)\,dt = f * K_T^\theta(x) \qquad (x\in\mathbb{R}^d, T\in\mathbb{R}_+^d),$$

where the Tth rectangular θ-kernel is given by

$$K_T^\theta(x) := \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \prod_{j=1}^d \theta_j\left(\frac{|t_j|}{T_j}\right) e^{\imath x\cdot t}\,dt = \frac{1}{(2\pi)^{d/2}} \prod_{j=1}^d T_j \widehat{\theta_j}(T_j x_j).$$

If $\widehat{\theta_j}\in L_1(\mathbb{R})$ $(j=1,\dots,d)$, then we can extend the definition of the rectangular θ-means to all $f\in W(L_1,\ell_\infty)(\mathbb{R}^d)$.

Theorem 4.6. *Assume that B is a homogeneous Banach space on $\mathbb{R}^d$. If θ_j satisfies (3.4) and $\widehat{\theta_j} \in L_1(\mathbb{R})$ for all $j = 1, \ldots, d$, then*

$$\left\|\sigma_T^\theta f\right\|_B \leq C \|f\|_B \qquad (f \in B, T \in \mathbb{R}_+^d)$$

and

$$\lim_{T\to\infty} \sigma_T^\theta f = f \qquad \text{in the } B\text{-norm for all } f \in B.$$

As a consequence, we obtain:

Corollary 4.3. *If f is a uniformly continuous and bounded function, θ_j satisfies (3.4) and $\widehat{\theta_j} \in L_1(\mathbb{R})$ for all $j = 1, \ldots, d$, then*

$$\lim_{T\to\infty} \sigma_T^\theta f = f \qquad \text{uniformly.}$$

5. ALMOST EVERYWHERE CONVERGENCE OF THE HIGHER DIMENSIONAL SUMMABILITY MEANS

As mentioned earlier, we will consider three subcases of the ℓ_0-summability, $q = 2$, $q = \infty$ and $q = 1$. Moreover, we also study two subcases of the rectangular summability, the restricted and the unrestricted summability. We will introduce for each case a new Hardy-Littlewood maximal function and a new type of Lebesgue points.

First, we generalize the $L_p(\mathbb{R}^d)$ spaces and introduce new function spaces. A measurable function f belongs to the Wiener amalgam space $W(L_p, \ell_q)(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$ if

$$\|f\|_{W(L_p,\ell_q)} := \left(\sum_{k\in\mathbb{Z}^d} \|f(\cdot + k)\|_{L_p[0,1)^d}^q\right)^{1/q} < \infty$$

if $1 \leq q < \infty$ and

$$\|f\|_{W(L_p,\ell_\infty)} := \left(\sup_{n_i\in\mathbb{Z}, i=1,\ldots,d} \int_{n_1}^{n_1+1} \cdots \int_{n_d}^{n_d+1} |f(x)|^p \, dx\right)^{1/p}$$

if $q = \infty$. It is easy to see that $W(L_p, \ell_p)(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ and

$$W(L_\infty, \ell_1)(\mathbb{R}^d) \subset L_p(\mathbb{R}^d) \subset W(L_1, \ell_\infty)(\mathbb{R}^d) \qquad (1 \leq p \leq \infty).$$

We modify slightly the definition of $W(L_p, \ell_\infty)(\mathbb{R}^d)$. If we change the integrals and the supremums in this expression, then we obtain the definition of the iterated Wiener amalgam spaces. In other words, a function f is in the iterated Wiener amalgam spaces $W_I(L_p, \ell_\infty)(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$ if

$$\|f\|_{W_I(L_p,\ell_\infty)} := \sup_{(i_1,\ldots,i_d)} \left(\sup_{n_{i_1}\in\mathbb{Z}} \int_{n_{i_1}}^{n_{i_1}+1} \cdots \sup_{n_{i_d}\in\mathbb{Z}} \int_{n_{i_d}}^{n_{i_d}+1} |f(x)|^p \, dx_{i_d} \cdots dx_{i_1}\right)^{1/p}$$

is finite. If we replace $|f(x)|^p$ by $|f(x)|^p (\log^+ |f(x)|)^k$ in the previous integral, then we get the definition of $W_I(L_p(\log L)^k, \ell_\infty)(\mathbb{R}^d)$ $(k \in \mathbb{N})$. Moreover, f is in the set $L_p(\log L)^k(\mathbb{R}^d)$ $(1 \leq p < \infty)$ if

$$\|f\|_{L_p(\log L)^k} := \left(\int_{\mathbb{R}^d} |f|^p (\log^+ |f|)^k \, d\lambda\right)^{1/p} < \infty,$$

where $\log^+ u := \max(0, \log u)$. It is easy to see that

$$W(L_p, \ell_\infty)(\mathbb{R}^d) \supset W_I(L_p(\log L)^k, \ell_\infty)(\mathbb{R}^d) \supset L_p(\log L)^k(\mathbb{R}^d), L_r(\mathbb{R}^d)$$

for all $1 \le p < r \le \infty$. Note that the space $W_I(L_p(\log L)^k, \ell_\infty)(\mathbb{R}^d)$ does not contain $L_p(\mathbb{R}^d)$.

5.1. **Circular summability.** For the circular summability, we need a new function space, the so called Herz space. We denote by $B(c,h)$ $(c \in \mathbb{R}^d, h > 0)$ the ball $\{x \in \mathbb{R}^d : \|x-c\|_2 < h\}$. Let the dyadic coronas be defined by

$$Q_k := B(0,2^k) \setminus B(0,2^{k-1}) \quad (k > 0), \qquad Q_0 := B(0,1).$$

The Herz space $E_q(\mathbb{R}^d)$ contains all measurable functions f for which

$$\|f\|_{E_q} := \sum_{k=0}^{\infty} 2^{dk(1-1/q)} \left\|f\mathbf{1}_{Q_k}\right\|_q < \infty.$$

Then obviously

$$L_1(\mathbb{R}^d) = E_1(\mathbb{R}^d) \supset E_q(\mathbb{R}^d) \supset E_{q'}(\mathbb{R}^d) \supset E_\infty(\mathbb{R}^d), \qquad 1 < q < q' < \infty.$$

Now we introduce a straightforward generalization of the Hardy-Littlewood maximal function by integrating on cubes:

$$M_p f(x) := \sup_{h>0} \left(\frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(x-t)|^p \, dt \right)^{1/p}.$$

A similar result holds as in the one-dimensional case (see e.g. Stein [30] and Weisz [41]).

Theorem 5.7. *For $1 \le p < \infty$,*

$$\sup_{\rho>0} \rho \lambda(M_p f > \rho)^{1/p} \le C\|f\|_p \qquad (f \in L_p(\mathbb{R}^d)),$$

$$\|M_p f\|_r \le C_r \|f\|_r \qquad (f \in L_r(\mathbb{R}^d), p < r \le \infty)$$

and

$$\|M_p f\|_{W(L_{p,\infty},\ell_\infty)} \le C\|f\|_{W(L_p,\ell_\infty)} \qquad (f \in W(L_p,\ell_\infty)(\mathbb{R}^d)),$$

$$\|M_p f\|_{W(L_r,\ell_\infty)} \le C_r \|f\|_{W(L_r,\ell_\infty)} \qquad (f \in W(L_r,\ell_\infty)(\mathbb{R}^d), p < r \le \infty).$$

The third inequality of the theorem implies:

Corollary 5.4. *If $f \in W(L_1,\ell_\infty)(\mathbb{R}^d)$, then*

$$\lim_{h\to 0} \frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} f(x-t)\, dt = f(x) \qquad a.e.\ x \in \mathbb{R}^d.$$

In other words,

$$\lim_{h\to 0} \frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} f(x-t) - f(x)\, dt = 0 \qquad \text{a.e. } x \in \mathbb{R}^d.$$

A point $x \in \mathbb{R}^d$ is called a p-Lebesgue point of f $(1 \le p < \infty)$ if

$$\lim_{h\to 0} \left(\frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(x-t) - f(x)|^p \, dt \right)^{1/p} = 0.$$

We can check easily that all r-Lebesgue points are p-Lebesgue points, whenever $p < r$. The definition of a p-Lebesgue point is depending only on f and every continuity point is a p-Lebesgue point. The following theorem can be found in Butzer and Nessel [6], Stein and Weiss [31] or Feichtinger and Weisz [9, 10].

Theorem 5.8. *Almost every point* $x \in \mathbb{R}^d$ *is a* p*-Lebesgue point of* $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$ *if* $1 \le p < \infty$.

Recall that $L_p(\mathbb{R}^d) \subset W(L_p, \ell_\infty)(\mathbb{R}^d)$. The first generalization of Lebesgue's theorem reads as follows.

Theorem 5.9. *Let* $\theta_0 \in L_1(\mathbb{R}^d)$, $1 \le p < \infty$ *and* $1/p + 1/q = 1$. *If* $\widehat{\theta}_0 \in E_q(\mathbb{R}^d)$, *then*

$$\lim_{T\to\infty} \sigma_T^{2,\theta} f(x) = f(x)$$

for all p*-Lebesgue points of* $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$.

The theorem is due to Feichtinger and Weisz [10]. Originally, it was proved for Riesz summation, for $p = 1$ and for integrable functions without using the Herz spaces in Stein and Weiss [31] or Butzer and Nessel [6]. We proved in [10] that the converse of the theorem holds also.

Theorem 5.10. *Suppose that* $\theta_0 \in L_1(\mathbb{R}^d)$, $\widehat{\theta}_0 \in L_1(\mathbb{R}^d)$, $1 \le p < \infty$ *and* $1/p + 1/q = 1$. *If*

$$\lim_{T\to\infty} \sigma_T^{2,\theta} f(x) = f(x)$$

for all p*-Lebesgue points of* $f \in L_p(\mathbb{R}^d)$, *then* $\widehat{\theta}_0 \in E_q(\mathbb{R}^d)$.

5.2. **Unrestricted rectangular summability.** Here we study another Hardy-Littlewood maximal function where we integrate on rectangles and not on cubes as before. The strong Hardy-Littlewood maximal function is defined by

$$M_{s,p} f(x) := \sup_{h_1,\ldots,h_d>0} \left(\frac{1}{2^d h_1 \cdots h_d} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-t)|^p \, dt \right)^{1/p}.$$

The next theorem was proved by the author [41].

Theorem 5.11. *Assume that* $1 \le p < \infty$ *and* $I = I_1 \times \cdots \times I_d$ *with* $|I_1| = \cdots = |I_d| = 1$. *If* $f \in L_p(\log L)^{d-1}(\mathbb{R}^d)$, *then*

$$\sup_{\rho>0} \rho \lambda(x : M_{s,p} f(x) > \rho, x \in I)^{1/p} \le C_p + C_p \|f\|_{L_p(\log L)^{d-1}}.$$

For $p < r \le \infty$,

$$\|M_{s,p} f\|_r \le C_r \|f\|_r \qquad (f \in L_r(\mathbb{R}^d)).$$

If $f \in W_I(L_p(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d)$, *then*

$$\|M_{s,p} f\|_{W(L_{p,\infty},\ell_\infty)} \le C_p + C_p \|f\|_{W_I(L_p(\log L)^{d-1},\ell_\infty)}$$

and, for $p < r \le \infty$,

$$\|M_{s,p} f\|_{W(L_r,\ell_\infty)} \le C_r \|f\|_{W_I(L_r,\ell_\infty)} \qquad (f \in W_I(L_r, \ell_\infty)(\mathbb{R}^d)).$$

Corollary 5.5. *If* $f \in W_I(L_1(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d)$, *then*

$$\lim_{h\to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} f(x-t)\, dt = f(x) \qquad \text{a.e. } x \in \mathbb{R}^d.$$

A point $x \in \mathbb{T}^d$ is called a strong p-Lebesgue point of f $(1 \le p < \infty)$ if

$$\lim_{h\to 0} \left(\frac{1}{2^d h_1 \cdots h_d} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-t) - f(x)|^p \, dt \right)^{1/p} = 0.$$

Here $h \to 0$ means that $h_i \to 0$ for all $i = 1, \ldots, d$. Again, every continuity point is a strong p-Lebesgue point.

The following two theorems are due to the author [39].

Theorem 5.12. *For* $1 \le p < \infty$*, almost every point* $x \in \mathbb{R}^d$ *is a strong* p*-Lebesgue point of* $f \in W_I(L_p(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d)$*.*

Recall that $\theta = \theta_1 \otimes \cdots \otimes \theta_d$ satisfies (3.4). It is easy to see that $\widehat{\theta} \in E_q(\mathbb{R}^d)$ if and only if $\widehat{\theta_j} \in E_q(\mathbb{R})$ for all $j = 1, \ldots, d$.

Theorem 5.13. *Let* $\theta \in L_1(\mathbb{R}^d)$*,* $1 \le p < \infty, 1/p + 1/q = 1$ *and* $\widehat{\theta} \in E_q(\mathbb{R}^d)$*. If* $f \in W_I(L_p(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d)$*,* x *is a strong* p*-Lebesgue point of* f *and* $M_{s,p}f(x)$ *is finite, then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = f(x).$$

Obviously, the convergence holds almost everywhere. Here $T \to \infty$ means again that $T_j \to \infty$ for all $j = 1, \ldots, d$. The iterated Wiener amalgam space $W_I(L_p(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d)$ is the largest function space for which these two theorems hold, even if we consider only almost everywhere convergence. So they are not true either for $W(L_p, \ell_\infty)(\mathbb{R}^d)$ or for $L_p(\mathbb{R}^d)$ (see Gát [12]). Now we formulate the converse of the preceding theorem.

Theorem 5.14. *Suppose that* $\theta \in L_1(\mathbb{R}^d)$*,* $\widehat{\theta} \in L_1(\mathbb{R}^d)$*,* $1 \le p < \infty$ *and* $1/p + 1/q = 1$*. If*

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = f(x)$$

for all strong p*-Lebesgue points of* $f \in L_p(\mathbb{R}^d)$*, then* $\widehat{\theta} \in E_q(\mathbb{R}^d)$*.*

5.3. **Restricted rectangular summability.** The third generalization is almost the same as the second one, the difference is that we assume here that T is in a cone (see Figure 2). For a given $\tau \ge 1$, we define a cone by

$$\mathbb{R}_\tau^d := \{x \in \mathbb{R}_+^d : \tau^{-1} \le x_i/x_j \le \tau, i, j = 1, \ldots, d\}.$$

The choice $\omega = 1$ obviously yields the diagonal.

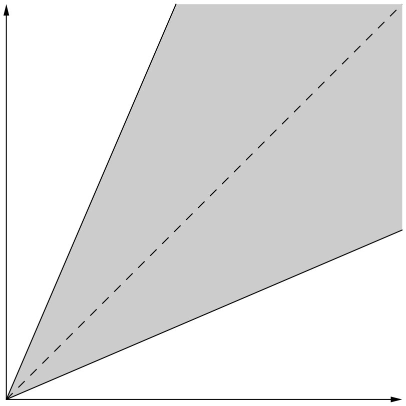

FIGURE 2. The cone for $d = 2$.

Instead of the Herz spaces we use here a weighted version of these spaces. The weighted Herz space $E_q^\omega(\mathbb{R})$ $(\omega \ge 0)$ contains all measurable functions f for which

$$\|f\|_{E_q^\omega} := \sum_{k=0}^{\infty} 2^{k(\omega+1-1/q)} \left\|f \mathbf{1}_{Q_k}\right\|_q < \infty.$$

Obviously,

$$E_q(\mathbb{R}) = E_q^0(\mathbb{R}) \supset E_q^\omega(\mathbb{R}) \qquad 0 \le \omega < \infty$$

and

$$L_1(\mathbb{R}) \supset E_1^\omega(\mathbb{R}) \supset E_q^\omega(\mathbb{R}) \supset E_{q'}^\omega(\mathbb{R}) \supset E_\infty^\omega(\mathbb{R}), \qquad 1 < q < q' < \infty.$$

Here, we cannot use the maximal functions introduced earlier, so we have to introduce a third generalization of the maximal function. For $\omega > 0$ and $1 \le p < \infty$, the Hardy-Littlewood maximal function $\mathcal{M}_p^{\omega,1} f$ is given by

$$\mathcal{M}_p^{\omega,1} f(x) := \sup_{i \in \mathbb{N}^d, h>0} 2^{-\omega\|i\|_1} \left(\frac{1}{(2h)^d 2^{\|i\|_1}} \int_{-2^{i_1}h}^{2^{i_1}h} \cdots \int_{-2^{i_d}h}^{2^{i_d}h} |f(x-t)|^p \, dt \right)^{1/p}.$$

If $\omega = 0$, we get back the definition of the strong Hardy-Littlewood maximal function $M_{s,p}f$. In contrary to the strong maximal function, due to the weight $2^{-\omega\|i\|_1}$, the weak type (p,p) inequality will be true for $\mathcal{M}_p^{\omega,1}$ (see Weisz [41]).

Theorem 5.15. *For $1 \le p < \infty$ and $\omega > 0$,*

$$\sup_{\rho>0} \rho \lambda(\mathcal{M}_p^{\omega,1} f > \rho)^{1/p} \le C\|f\|_p \qquad (f \in L_p(\mathbb{R}^d)),$$

$$\left\|\mathcal{M}_p^{\omega,1} f\right\|_r \le C_r \|f\|_r \qquad (f \in L_r(\mathbb{R}^d), p < r \le \infty)$$

and

$$\left\|\mathcal{M}_p^{\omega,1} f\right\|_{W(L_{p,\infty},\ell_\infty)} \le C\|f\|_{W(L_p,\ell_\infty)} \qquad (f \in W(L_p,\ell_\infty)(\mathbb{R}^d)),$$

$$\left\|\mathcal{M}_p^{\omega,1} f\right\|_{W(L_r,\ell_\infty)} \le C_r\|f\|_{W(L_r,\ell_\infty)} \qquad (f \in W(L_r,\ell_\infty)(\mathbb{R}^d), p < r \le \infty).$$

Note that the definitions of the p-Lebesgue points and strong p-Lebesgue points can be rewritten as

$$\lim_{r\to 0} \sup_{0<h<r} \left(\frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(x-t) - f(x)|^p \, dt \right)^{1/p} = 0$$

and

$$\lim_{r\to 0} \sup_{0<h_j<r, j=1,\ldots,d} \left(\frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-t) - f(x)|^p \, dt \right)^{1/p} = 0,$$

respectively. Similarly to this definition, we introduce a new type of Lebesgue points. For $1 \le p < \infty$ and $\omega > 0$, a point $x \in \mathbb{R}^d$ is called a (p,ω)-Lebesgue point of f if

$$\lim_{r\to 0} \sup_{i\in\mathbb{N}^d, h>0, 2^{i_k}h<r, k=1,\ldots d} 2^{-\omega\|i\|_1} \left(\frac{1}{(2h)^d 2^{\|i\|_1}} \int_{-2^{i_1}h}^{2^{i_1}h} \cdots \int_{-2^{i_d}h}^{2^{i_d}h} |f(x-t) - f(x)|^p \, dt \right)^{1/p} = 0.$$

If $\omega = 0$, then the $(p,0)$-Lebesgue points are the same as the strong p-Lebesgue points. If f is continuous at x, then x is a (p,ω)-Lebesgue point of f for all $1 \le p < \infty$ and $\omega > 0$. The next two theorems are due to the author [42]. A first version of Theorem 5.17 was shown by Marcinkiewicz and Zygmund [25] in 1939. Later Gát [12] and the author [35, 44] proved the almost everywhere convergence.

Theorem 5.16. *For $1 \le p < \infty$, almost every point $x \in \mathbb{R}^d$ is a (p,ω)-Lebesgue point of $f \in W(L_p,\ell_\infty)(\mathbb{R}^d)$.*

Theorem 5.17. *Let* $\theta_i \in L_1(\mathbb{R})$, $1 \leq p < \infty$, $1/p + 1/q = 1$, $\omega > 0$ *and* $\widehat{\theta}_i \in E_q^{\omega}(\mathbb{R})$ $(i = 1, \ldots, d)$. *If* $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$, x *is a* (p, ω)*-Lebesgue point of* f *and* $\mathcal{M}_p^{\omega,1} f(x)$ *is finite, then*

$$\lim_{T\to\infty, T\in\mathbb{R}_\tau^d} \sigma_T^{\theta} f(x) = f(x).$$

Of course, the convergence holds almost everywhere.

5.4. **Cubic summability.** In the fourth generalization, when $q = \infty$, we do not use Herz spaces. Instead, we will use conditions (4.6) and (4.7).

To introduce the next Hardy-Littlewood maximal function, let us denote by $P_{2^{i_1}h,\ldots,2^{i_d}h}$ a parallelepiped, whose center is the origin and whose sides are parallel to the axes and/or to the diagonals and whose kth side length is $2^{i_k+1}h$ if the kth side is parallel to an axis and $\sqrt{2}2^{i_k+1}h$ if the kth side is parallel to a diagonal $(i \in \mathbb{N}^d, h > 0, k = 1, \ldots, d)$. More exactly, at least one side of $P_{2^{i_1}h,\ldots,2^{i_d}h}$ is parallel to one of the axes and the other sides are parallel to the axes and/or to the diagonals.

For $\omega > 0$ and $1 \leq p < \infty$, the Hardy-Littlewood maximal function $\mathcal{M}_p^{\omega} f$ is given by

$$\mathcal{M}_p^{\omega} f(x) := \sup_{P_{2^{i_1}h,\ldots,2^{i_d}h}, i\in\mathbb{N}^d, h>0} 2^{-\omega\|i\|_1} \left(\frac{1}{|P_{2^{i_1}h,\ldots,2^{i_d}h}|} \int_{P_{2^{i_1}h,\ldots,2^{i_d}h}} |f(x-t)|^p \, dt \right)^{1/p},$$

where the supremum is taken over all parallelepipeds $P_{2^{i_1}h,\ldots,2^{i_d}h}$ $(i \in \mathbb{N}^d, h > 0)$ just defined. If we take the supremum only over all rectangles with sides parallel to the axes, we get back the definition of the maximal operator $\mathcal{M}_p^{\omega,1} f$. 5.15 remains true for the maximal operator $\mathcal{M}_p^{\omega}$ if $1 \leq p < \infty$ and $\omega > 0$.

For $1 \leq p < \infty$ and $\omega > 0$, a point $x \in \mathbb{R}^d$ is called a strong (p, ω)-Lebesgue point of f if

$$\lim_{r\to 0} \sup_{\substack{P_{2^{i_1}h,\ldots,2^{i_d}h}, i\in\mathbb{N}^d, h>0 \\ 2^{i_k}h<r, k=1,\ldots d}} 2^{-\omega\|i\|_1} \left(\frac{1}{|P_{2^{i_1}h,\ldots,2^{i_d}h}|} \right)^{1/p} \left(\int_{P_{2^{i_1}h,\ldots,2^{i_d}h}} |f(x-t) - f(x)|^p \, dt \right)^{1/p} = 0.$$

The next theorems of this subsection were proved in [40].

Theorem 5.18. *Almost every point* $x \in \mathbb{R}^d$ *is a strong* (p, ω)*-Lebesgue point of* $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$ *if* $1 \leq p < \infty$.

If $p > 1$ and $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$, then we do not need the concept of strong (p, ω)-Lebesgue points just introduced, it is enough to consider (p, ω)-Lebesgue points.

Theorem 5.19. *Suppose that* $1 < p < \infty$ *and the conditions* (4.6) *and* (4.7) *are satisfied. If* $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$, x *is a* (p, ω)*-Lebesgue point of* f *and* $\mathcal{M}_p^{\omega,1} f(x)$ *is finite, then*

$$\lim_{T\to\infty} \sigma_T^{\infty,\theta} f(x) = f(x).$$

This result does not hold for $p = 1$. In this case, we need the concept of strong (p, ω)-Lebesgue points.

Theorem 5.20. *Suppose that the conditions* (4.6) *and* (4.7) *are satisfied. If* $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$, x *is a strong* $(1, \omega)$*-Lebesgue point of* f *and* $\mathcal{M}_1^{\omega} f(x)$ *is finite, then*

$$\lim_{T\to\infty} \sigma_T^{\infty,\theta} f(x) = f(x).$$

Using the theorems of this section, we have given simple proofs for the classical strong summability results in [40].

5.5. **Triangular summability.** The convergence results are similar to those for cubic summability and are proved in [43]. We use the same Hardy-Littlewood maximal function and the same type of Lebesgue points as in Subsection 5.4.

Theorem 5.21. *Suppose that* $1 < p < \infty$ *and the conditions* (4.6) *and* (4.7) *are satisfied. If* $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$*,* x *is a* (p,ω)*-Lebesgue point of* f *and* $\mathcal{M}_p^{\omega,1} f(x)$ *is finite, then*

$$\lim_{T\to\infty} \sigma_T^{1,\theta} f(x) = f(x).$$

Theorem 5.22. *Suppose that the conditions* (4.6) *and* (4.7) *are satisfied. If* $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$*,* x *is a strong* $(1,\omega)$*-Lebesgue point of* f *and* $\mathcal{M}_1^{\omega} f(x)$ *is finite, then*

$$\lim_{T\to\infty} \sigma_T^{1,\theta} f(x) = f(x).$$

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