ALTAY CONFERENCE PROCEEDINGS IN MATHEMATICS
2 (2025), No. 1, pp. 48-59
`https://altayconfproceedings.com/`
ISSN 3108 - 5288

Research Article

Approximation characteristics of Stepanets-Orlicz type spaces

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ABSTRACT. In the paper, we introduce spaces $\mathcal{S}_{M,\Phi}$ and show their connection with the well-known Stepanets spaces $\mathcal{S}^p_\Phi$, Orlicz spaces $\mathcal{S}_M$, and others. We obtain exact upper bounds for quantities analogous to trigonometric widths and best n-term approximations of certain compact sets in these spaces.

Keywords: Stepanets spaces, Orlicz spaces, best approximation, best n-term approximation.

2020 Mathematics Subject Classification: 41A25, 41A65.

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1. INTRODUCTION

In 2003, Stepanets [12] introduced the spaces $\mathcal{S}^p_\Phi = \mathcal{S}^p_\Phi(\mathscr{X}, \mathscr{Y})$. His approach made it possible to consider the usual problems of approximation theory in general linear spaces, and at the same time, by considering special cases of his constructions, it is possible to obtain quite meaningful results in many well-known spaces. Here, we consider the spaces $\mathcal{S}_{M,\Phi}$, which constructed similarly to the spaces $\mathcal{S}^p_\Phi$ and generated by a certain Orlicz function $M(u)$, $u \geq 0$. In the case when M is a power function, i.e., $M(u) = u^p$, $p \geq 1$, the spaces $\mathcal{S}_{M,\Phi}$ coincide with the spaces $\mathcal{S}^p_\Phi$. However, this construction also makes it possible to obtain, as special cases of spaces $\mathcal{S}_{M,\Phi}$, some other important spaces of the Orlicz type.

2. THE SPACES $\mathcal{S}_{M,\Phi}$

2.1. **Definition of the spaces $\mathcal{S}_{M,\Phi}$.** Let us introduce spaces $\mathcal{S}_{M,\Phi}$ and show their connection with other well-known spaces. In defining them and further in the paper, we will mainly use the symbols and definitions proposed in [12].

Let $\mathscr{X}$ and $\mathscr{Y}$ be some linear spaces of elements x and y, respectively. Suppose that a linear operator Φ is defined on $\mathscr{X}$ acting in $\mathscr{Y}$, and a functional f is defined on some subset $\mathscr{Y}' \subset \mathscr{Y}$. Let $E(\Phi)$ be the range of the operator Φ, and $\mathscr{X}'$ be the preimage of the set $\mathscr{Y}' \subset E(\Phi)$ under the mapping Φ. In this case, on $\mathscr{X}'$ we can define the functional f' using the equality

$$f'(x) = f(\Phi(x)), \qquad x \in \mathscr{X}' \tag{2.1}$$

If we choose as f a functional that defines a norm (or quasi-norm) on $\mathscr{Y}'$, then equality (2.1) will define a similar quantity on $\mathscr{X}'$. These considerations form the basis for further constructions.

Let $(\mathbb{R}^d, \mathfrak{B}, \mathrm{d}\mu)$, $d \geq 1$, be a d-dimensional Euclidean space of points $\mathbf{t} = (t_1, \ldots, t_d)$, defined on a Borel σ-algebra $\mathfrak{B}$ of subsets of $\mathbb{R}^d$, with a finite σ-additive continuous measure $\mathrm{d}\mu$. Let

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Received: 03.07.2025; Accepted: 31.10.2025;

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DOI: 10.64700/altay.24
Presented in *International Workshop on Modern Problems of Analysis, Optimization, Approximation and Their Applications*

$\mathbb{A}$ be a μ-dimensional subset of $(\mathbb{R}^d, \mathfrak{B}, \mathrm{d}\mu)$, whose μ-measure is equal to a, where a is a finite number, or $a = +\infty$:

$$\mu(\mathbb{A}) = |\mathbb{A}|_\mu = a, \quad a \in (0, +\infty].$$

Let also $\mathscr{Y} = \mathscr{Y}(\mathbb{A}, \mathrm{d}\mu)$ be the set of all functions $f = f(\mathbf{t})$ defined on $\mathbb{A}$ and measurable with respect to the measure $\mathrm{d}\mu$.

Next, let $M(u)$, $u \geq 0$, be an arbitrary Orlicz function, i.e., a non-decreasing convex function such that $M(0) = 0$ and $M(u) \to +\infty$ as $u \to +\infty$. Denote by $L_M(\mathbb{A}, \mathrm{d}\mu)$ the set of all functions $f \in \mathscr{Y}(\mathbb{A}, \mathrm{d}\mu)$ that satisfy the condition

$$(\forall C > 0) \qquad \int_{\mathbb{A}} M(C|f(\mathbf{t})|)\, \mathrm{d}\mu(\mathbf{t}) < +\infty.$$

The linear space $L_M(\mathbb{A}, \mathrm{d}\mu)$ is a Banach space with the Luxembourg norm

$$\|f\|_{L_M(\mathbb{A},\, \mathrm{d}\mu)} := \inf \left\{ \alpha > 0 : \int_{\mathbb{A}} M\Big(\tfrac{|f(\mathbf{t})|}{\alpha}\Big)\, \mathrm{d}\mu(\mathbf{t}) \leq 1 \right\}, \tag{2.2}$$

and it is called the Orlicz space.

Note that in the case when $M(u) = u^p$, $p \geq 1$, the spaces $L_M(\mathbb{A}, \mathrm{d}\mu)$ coincide with the well-known Lebesgue spaces $L_p(\mathbb{A}, \mathrm{d}\mu)$, which consist of functions $f \in \mathscr{Y}(\mathbb{A}, \mathrm{d}\mu)$ such that

$$\|f\|_{L_p(\mathbb{A},\, \mathrm{d}\mu)} = \Big(\int_{\mathbb{A}} |f(\mathbf{t})|^p \mathrm{d}\mu(\mathbf{t}) \Big)^{\frac{1}{p}} < +\infty.$$

Let $\mathscr{X}$ be a linear space of elements x, and Φ be a linear operator acting from $\mathscr{X}$ to $\mathscr{Y}(A, \mathrm{d}\mu)$:

$$\Phi : \mathscr{X} \to \mathscr{Y}(\mathbb{A},\, \mathrm{d}\mu), \quad \Phi(x) =: \widehat{x}, \quad x \in \mathscr{X}, \quad \widehat{x} \in \mathscr{Y}(\mathbb{A},\, \mathrm{d}\mu). \tag{2.3}$$

We set

$$\mathcal{S}_{M,\Phi} = \mathcal{S}_{M,\Phi}(\mathscr{X}, \mathscr{Y}) = \Big\{ x \in \mathscr{X} : \|\widehat{x}\|_{L_M(\mathbb{A},\, \mathrm{d}\mu)} < +\infty \Big\}. \tag{2.4}$$

Thus, the set $\mathcal{S}_{M,\Phi}$ is the set of all elements $x \in \mathscr{X}$ that are preimages of functions from the set $L_M(\mathbb{A},\, \mathrm{d}\mu)$ under the mapping Φ.

Elements $x_1, x_2 \in \mathscr{X}$ are considered identical in $\mathcal{S}_{M,\Phi}$ if, with respect to measure $\mathrm{d}\mu$, $\widehat{x}_1(\mathbf{t}) = \widehat{x}_2(\mathbf{t})$ almost everywhere on $\mathbb{A}$. For elements $x_1, x_2 \in \mathcal{S}_{M,\Phi}$, the Φ-distance between them is defined by the equality

$$\rho_\Phi(x_1, x_2)_{M,\Phi} = \|\Phi(x_1 - x_2)\|_{L_M(\mathbb{A},\, \mathrm{d}\mu)}.$$

An element θ is called a zero element of the set $\mathcal{S}_{M,\Phi}$ if the equality $\widehat{\theta}(\mathbf{t}) = 0$ holds almost everywhere on $\mathbb{A}$.

The distance $\rho_\Phi(\theta, x)_{M,\Phi}$ is called the Φ-norm of the element $x \in \mathcal{S}_{M,\Phi}$ and it is denoted by $\|x\|_{M,\Phi}$. Thus, by definition

$$\|x\|_{M,\Phi} = \rho_\Phi(\theta, x)_{M,\Phi} = \|\widehat{x}\|_{L_M(\mathbb{A},\, \mathrm{d}\mu)}. \tag{2.5}$$

In the terminology accepted in the theory of integral transformations, the element $\widehat{x} = \Phi(x)$ is the image (Φ-image) of an element x, and the set $E(\Phi)$ of values of the operator Φ is the set of images. Thus, the Φ-distance and Φ-norm are the distance and norm in the space of images.

2.2. **Examples of the spaces $\mathcal{S}_{M,\Phi}$.**

Let us consider several examples of the simplest implementations of the constructions under consideration. Here, we will say that a certain space $\mathfrak{N}$ is a special case of the space $\mathcal{S}_{M,\Phi}$ if it can be obtained by appropriately choosing the space $\mathscr{X}$, the measure $\mathrm{d}\mu$, and the operator Φ.

2.2.1. *Spaces* $\mathcal{S}^p_\Phi$. In the case when $M(u) = t^p$, $p \geq 1$, the spaces $\mathcal{S}_{M,\Phi}$ coincide with the Stepanets spaces $\mathcal{S}^p_\Phi$ [12]. Note that in [12], many special cases in the above sense of spaces $\mathcal{S}^p_\Phi$ were also presented. Approximative characteristics of the spaces $\mathcal{S}^p_\Phi$ were studied in [12, 13, 16, 8], etc.

2.2.2. *Spaces* $\mathcal{S}_{M,\varphi}$. Let $\mathscr{X}$ be a linear complex space, let $\varphi = \{\varphi_k\}_{k=1}^\infty$ be a fixed countable linearly independent system in this space, and let any pair $x, y \in \mathscr{X}$, in which at least one element belongs to φ, be associated with a number (x, y) such that the following conditions are satisfied:

1) $(x,y) = \overline{(y,x)}$, where $\overline{z}$ is the complex conjugate of z;
2) $(\lambda x_1 + \mu x_2, y) = \lambda(x_1, y) + \mu(x_2, y)$, where λ and μ are arbitrary numbers;
3) $(\varphi_k, \varphi_l) = \begin{cases} 0, & k \neq l; \\ 1, & k = l. \end{cases}$

Thus, the scalar product of elements of the space $\mathscr{X}$ by elements of the system φ is defined. Every element $x \in \mathscr{X}$ is associated with a system of numbers $\widehat{x}_\varphi$ such that

$$\widehat{x}_\varphi(k) = (x, \varphi_k), \qquad k = 1, 2, \ldots \qquad (k \in \mathbb{N})$$

and for arbitrary Orlicz function M, we consider the sets

$$\mathcal{S}_{M,\varphi} = \mathcal{S}_{M,\varphi}(\mathscr{X}) = \Big\{ x \in \mathscr{X} : \sum_{k=1}^\infty M\big(C\widehat{x}_\varphi(k)\big) < \infty, \quad \forall C > 0 \Big\}. \tag{2.6}$$

Elements $x, y \in \mathscr{X}$ are considered identical in $\mathcal{S}_{M,\varphi}$ if $\widehat{x}_\varphi(k) = \widehat{y}_\varphi(k)$ for all $k \in \mathbb{N}$.

For $x \in \mathcal{S}_{M,\varphi}$, the φ-norm of the element x is denoted by $\|x\|_{p,\varphi}$. Thus,

$$\|x\|_{M,\varphi} := \inf \Big\{ \alpha > 0 : \sum_{k=1}^\infty M\left(\tfrac{|\widehat{x}_\varphi(k)|}{\alpha}\right) \leq 1 \Big\}. \tag{2.7}$$

The spaces $\mathcal{S}_{M,\varphi}$ are a special case of the spaces $\mathcal{S}_{M,\Phi}$. Indeed, in the given space $\mathscr{X}$, we define an operator Φ that for every $x \in \mathscr{X}$ assigns the sequence $y = \{\widehat{x}_k\}_{k=1}^\infty$. We take the space $\mathbb{R}^1$ with measure $\mathrm{d}\mu$, whose support is the set $\mathbb{Z}^1$ of integer points k in which $\mu(k) \equiv 1$, and set $\mathbb{A} = \big\{ k \in \mathbb{Z}^1, \quad k \geq 1 \big\}$. In this case $\mathscr{Y}(\mathbb{A}, \mathrm{d}\mu)$ is the set of all sequences y, and the functional (2.2) has the form

$$\|y\|_{L_M(\mathbb{A},\mathrm{d}\mu)} = \inf \Big\{ \alpha > 0 : \sum_{k=1}^\infty M\big(\tfrac{|y_k|}{\alpha}\big) \leq 1 \Big\}. \tag{2.8}$$

2.2.3. *Spaces* $\mathcal{S}^\mu_{M,\varphi}$. These spaces are defined in the same way as the spaces S^p_φ, except that the functionals of the form

$$\sum_{k=1}^\infty M(\cdot)$$

in the equalities corresponding to equalities (2.6)–(2.8), are replaced by functionals of the form

$$\sum_{k=1}^\infty \mu_k M(\cdot),$$

where $\mu = \{\mu_k\}_{k=1}^\infty$ is a given system of non-negative numbers, $\mu_k \geq 0$, $k \in \mathbb{N}$. In this case, if $\mu_k \equiv 1$, then $\mathcal{S}^\mu_{M,\varphi} = \mathcal{S}_{M,\varphi}$.

It is clear that these spaces are a special case of the spaces $S_{M,\Phi}$. In this case, we similarly take the space $\mathbb{R}^1$ with measure $\mathrm{d}\mu$, whose support is the set $\mathbb{Z}^1$ of integer points k in which $\mu(k) = \mu_k$ and $\mathbb{A} = \{k \in \mathbb{Z}^1, \ k \geq 1\}$.

2.2.4. *Spaces* $\mathcal{S}_M(\mathbb{T}^d)=\mathcal{S}_{M,\mathcal{F}}(L_1(\mathbb{T}^d))$. Let, as above, $\mathbb{R}^d$ be an d-dimensional, $d\ge 1$, Euclidean space of vectors $\mathbf{t}=(t_1,\ldots,t_d)$ and $\mathbb{Z}^d$ be an integer lattice in $\mathbb{R}^d$. Denote by $\mathbb{T}^d:=[0,2\pi)^d$ a d-dimensional torus and set $(\mathbf{t},\mathbf{y}):=t_1y_1+\ldots+t_dy_d$, $\mathbf{t},\mathbf{y}\in\mathbb{R}^d$

Let, further, $L_1=L_1(\mathbb{T}^d)$ be the set of all Lebesgue-summable on $\mathbb{R}^d$ functions $f(\mathbf{t})=f(t_1,\ldots t_d)$, 2π-periodic in each variable. We take as $\mathscr{X}$ the space $L_1(\mathbb{T}^d)$ and define on it the operator Φ (which we will denote by $\mathcal{F}$), setting

$$\mathcal{F}(f)=(2\pi)^{-d}\int\limits_{\mathbb{T}^d} f(\mathbf{t})\,e^{-\mathrm{i}(\mathbf{k},\mathbf{t})}\mathrm{d}\mathbf{t}=\widehat{f}(\mathbf{k}),\quad \mathbf{k}\in\mathbb{Z}^m.$$

This operator maps the space $L_1(\mathbb{T}^d)$ to the set $\mathscr{Y}$ of functions y defined on the integer lattice $\mathbb{Z}^d$. Let $\mathrm{d}\mu$ be a measure in the space $\mathbb{R}^d$, whose support is the set $\mathbb{Z}^d$, where it equals one. In this case, the functional (2.2) has the form

$$\|f\|_{L_M(\mathbb{R}^d,\,\mathrm{d}\mu)}:=\inf\left\{\alpha>0:\int\limits_{\mathbb{R}^d} M\Big(\tfrac{|f(\mathbf{t})|}{\alpha}\Big)\,\mathrm{d}\mu(\mathbf{t})\le 1\right\}=\inf\left\{\alpha>0:\sum_{k\in\mathbb{Z}^d} M\Big(\tfrac{|\widehat{f}(\mathbf{k})|}{\alpha}\Big)\le 1\right\},$$

and the space $\mathcal{S}_{M,\Phi}$ (which we denote by $\mathcal{S}_M(\mathbb{T}^d)$) is defined by the relation

$$\mathcal{S}_M\left(\mathbb{T}^d\right)=\mathcal{S}_{M,\mathcal{F}}(L_1(\mathbb{T}^d))=\left\{f\in L_1(\mathbb{T}^d):\quad (\forall C>0)\sum_{k\in\mathbb{Z}^d} M\big(C|\widehat{f}(\mathbf{k})|\big)<+\infty\right\}.$$

Note that the spaces $\mathcal{S}_M(\mathbb{T}^1)$ were considered in [1, 2], where, in particular, direct and inverse approximation theorems in these spaces were proved. These spaces coincide with the spaces $S_{M,\varphi}(L)$ generated by the set $L_1(\mathbb{T}^d)$ and the system $\varphi=\{\tau_s\}_{s=1}^\infty$, $\tau_s=\tau_s(\mathbf{x})=\{\mathrm{e}^{\mathrm{i}(\mathbf{k}_s,\mathbf{x})}\}$, $\mathbf{k}_s\in\mathbb{Z}^d$, $s\in\mathbb{N}$, which can be obtained by arbitrarily numbering the elements of the system $\{\mathrm{e}^{\mathrm{i}(\mathbf{k},\mathbf{x})}\}_{\mathbf{k}\in\mathbb{Z}^d}$.

In the case when $M(u)=u^p$, $p\ge 1$, the spaces $\mathcal{S}_{M,\varphi}$ are the spaces $\mathcal{S}^p_\varphi$, the spaces $\mathcal{S}^\mu_{M,\varphi}$ are the spaces $\mathcal{S}^{p,\mu}_\varphi$, and the spaces $\mathcal{S}_M(\mathbb{T}^d)$ are the Wiener type spaces $\mathcal{S}^p(\mathbb{T}^d)$ [11], which were studied by many authors (see, for example [5, 15, 14, 13, 6, 7], and references therein).

3. MULTIPLIERS. APPROXIMATING AGGREGATES AND OBJECTS OF APPROXIMATION

Approximating aggregates for elements $x\in\mathcal{S}_{M,\Phi}$ and objects of approximation are defined similarly to [12]. We use elements from $\mathcal{S}_{M,\Phi}$ whose images have supports γ_σ of given measure σ. It is clear that exactly this principle is used in the classical case in the construction of, e.g., trigonometric polynomials for the approximation of a given periodic function if the operator Φ is understood as the mapping of functions into the set of their Fourier coefficients. In the general case, there arise certain problems related to the fact that the spaces $\mathcal{S}_{M,\Phi}$ can be incomplete. In this connection, we introduce the following definitions.

Let $\omega=\omega(\mathbf{t})$ be a certain function from $\mathscr{Y}(\mathbb{A},\mathrm{d}\mu)$. Then we denote by $\mathcal{M}^\omega_\Phi$ the operator acting from $\mathscr{X}$ into $\mathscr{X}$ that associates $x\in\mathscr{X}$ with an element $x_\omega\in\mathscr{X}$ such that if $\Phi(x)=\widehat{x}$, then $\widehat{x}_\omega(\mathbf{t})=\Phi(x_\omega)=\omega(\mathbf{t})\widehat{x}(\mathbf{t})$ almost everywhere on $\mathbb{A}$. The operator $\mathcal{M}^\omega_\Phi$ is called the multiplier of the operator Φ generated by the function ω. Let $\Omega_\Phi(\mathscr{X})=\Omega_\Phi(\mathscr{X},\mathscr{X})$ denote the subset of functions ω from $\mathscr{Y}(\mathbb{A},\mathrm{d}\mu)$ for which the multipliers $\mathcal{M}^\omega_\Phi$ exist.

If $\mathfrak{N}$ and $\mathfrak{N}'$ are some subsets of $\mathscr{X}$, $\omega\in\Omega_\Phi(\mathscr{X})$, and the operator $\mathcal{M}^\omega_\Phi$ maps $\mathfrak{N}$ into $\mathfrak{N}'$, then we say that $\mathcal{M}^\omega_\Phi$ has the type $(\mathfrak{N},\mathfrak{N}')$. In particular, if $\mathcal{M}^\omega_\Phi$ maps $\mathcal{S}_{M,\Phi}$ into $\mathcal{S}_{M,\Phi}$, then the operator $\mathcal{M}^\omega_\Phi$ has the type $(\mathcal{S}_{M,\Phi},\mathcal{S}_{M,\Phi})$, or, briefly, the type (M,M). The set of functions ω generating operators of the type (M,M) is denoted by $\Omega_{M,\Phi}$. Thus, if $\omega\in\Omega_{M,\Phi}$ and the operator $\mathcal{M}_\Phi^\omega$ acts from $\mathcal{S}_{M,\Phi}$, then it acts into $\mathcal{S}_{M,\Phi}$. In this case, every $x \in \mathcal{S}_{M,\Phi}$ is associated with an element $x_\omega = \mathcal{M}_\Phi^\omega(x)$ for which the following equality holds almost everywhere on $\mathbb{A}$:

$$\widehat{x}_\omega(\mathbf{t}) = \Phi(x_\omega) = \omega(\mathbf{t})\widehat{x}(\mathbf{t}), \quad \widehat{x}_\omega \in L_M(\mathbb{A}, \mathrm{d}\mu). \tag{3.9}$$

Given $\sigma > 0$, assume that γ_σ is a μ-measurable set in $\mathbb{A}$, $\mathrm{mes}_\mu \gamma_\sigma =: |\gamma_\sigma| = \sigma$, $\sigma \le a$, and $\lambda = \lambda(\mathbf{t})$ is a measurable function with support γ_σ. Also assume that, for a given Orlicz function M, we have $\lambda \in \Omega_{M,\Phi}$ and $U_{\gamma_\sigma}(x, \lambda) := x_\lambda = \mathcal{M}_\Phi^\lambda(x)$. According to (3.9), we get

$$\widehat{U}_{\gamma_\sigma}(x,\lambda) = \Phi(U_{\gamma_\sigma}(x,\lambda)) = \begin{cases} \lambda(\mathbf{t})\widehat{x}(\mathbf{t}), & \mathbf{t} \in \gamma_\sigma, \\ 0, & \mathbf{t} \bar{\in} \gamma_\sigma, \end{cases} \qquad x \in \mathcal{S}_{M,\Phi}. \tag{3.10}$$

The elements $U_{\gamma_\sigma}(x,\lambda)$ are considered as approximating aggregates for $x \in \mathcal{S}_{M,\Phi}$. In this case, if $\lambda(\mathbf{t}) \equiv 1$ on γ_σ, i.e., if $\lambda(\mathbf{t})$ coincides with the characteristic function $\chi_{\gamma_\sigma}(\mathbf{t})$ of the set γ_σ, then we set $U_{\gamma_\sigma}(x, \chi_{\gamma_\sigma}) =: U_{\gamma_\sigma}(x)$.

Let $\Gamma_\sigma = \Gamma_\sigma(\mathbb{A})$ be the set of all measurable subsets of $\mathbb{A}$ whose measures are equal to σ. We say that, for a given Orlicz function M, an operator Φ satisfies condition (A_M) if the functions $\chi_{\gamma_\sigma}(\mathbf{t})$ of all sets $\gamma_\sigma \in \Gamma_\sigma$ belong to $\Omega_{M,\Phi}$ for all $\sigma \in [0, a)$. Thus, if Φ satisfies condition (A_M), then all elements $U_{\gamma_\sigma}(x)$ are defined for any $x \in \mathcal{S}_{M,\Phi}$ and are contained in $\mathcal{S}_{M,\Phi}$. The element $U_{\gamma_\sigma}(x)$ is called the restriction of an element x of rank σ, and the element $U_{\gamma_\sigma}(x,\lambda)$ is called the λ-restriction of x of the rank σ.

Let M be any Orlicz function and let $x \in \mathcal{S}_{M,\Phi}$. Then, by virtue of (2.5) and (3.10), we get

$$\begin{aligned} &\|x - U_{\gamma_\sigma}(x,\lambda)\|_{M,\Phi} \\ =&\|\widehat{x} - \widehat{U}_{\gamma_\sigma}(x,\lambda)\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)} \\ =&\inf\left\{\alpha > 0 : \int_{\gamma_\sigma} M\Big(\tfrac{|1-\lambda(\mathbf{t})||\widehat{x}(\mathbf{t})|}{\alpha}\Big)\, d\mu(\mathbf{t}) + \int_{\mathbb{A}\setminus\gamma_\sigma} M\Big(\tfrac{|\widehat{x}(\mathbf{t})|}{\alpha}\Big)\mathrm{d}\mu(\mathbf{t}) \le 1\right\}. \end{aligned}$$

Hence, we arrive at the following statement:

Proposition 3.1. *Suppose that M is any Orlicz function, $x \in \mathcal{S}_{M,\Phi} = \mathcal{S}_{M,\Phi}(\mathscr{X}, \mathscr{Y})$, $\gamma_\sigma \in \Gamma_\sigma$ and the operator Φ satisfies condition (A_M). Then*

$$\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi} := \inf_{\lambda \in \Omega_{M,\Phi}} \| x - U_{\gamma_\sigma}(x,\lambda)\|_{M,\Phi} = \| x - U_{\gamma_\sigma}(x)\|_{M,\Phi}.$$

Furthermore, the following equality is true:

$$\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi} = \inf\left\{\alpha > 0 : \int_{\mathbb{A}\setminus\gamma_\sigma} M\Big(\tfrac{|\widehat{x}(\mathbf{t})|}{\alpha}\Big)\mathrm{d}\mu(\mathbf{t}) \le 1\right\}. \tag{3.11}$$

Thus, if $\chi_{\gamma_\sigma} \in \Omega_{M,\Phi}$, among all elements $U_{\gamma_\sigma}(x,\lambda)$ generated by the multipliers $\mathcal{M}_\Phi^\lambda$ and satisfying condition (3.10), the element $U_{\gamma_\sigma}(x)$ has the least deviation from an element x in Φ-norm in the space $\mathcal{S}_{M,\Phi}$, i.e., among all λ-restrictions of x of given rank σ, its restriction for $\lambda(\mathbf{t}) \equiv 1$ is the closest one to x. It is clear that this property is an analog of the minimum property of Fourier sums in the Hilbert spaces.

Let $\Gamma = \{\gamma_\sigma\}_{\sigma>0}$, $|\gamma_\sigma| = \sigma$, be a family of measurable subsets of $\mathbb{A}$ that exhausts the entire set $\mathbb{A}$ for $\sigma \to +\infty$, i.e., it possesses the property that any point $\mathbf{t} \in \mathbb{A}$ is contained in all sets γ_σ for all sufficiently large values of σ and, therefore, for any $\alpha > 0$,

$$\lim_{\sigma\to+\infty} \int_{\gamma_\sigma} M\Big(\tfrac{|\widehat{x}(\mathbf{t})|}{\alpha}\Big)\mathrm{d}\mu(\mathbf{t}) = \int_{\mathbb{A}} M\Big(\tfrac{|\widehat{x}(\mathbf{t})|}{\alpha}\Big)\mathrm{d}\mu(\mathbf{t}), \qquad \forall x \in \mathcal{S}_{M,\Phi}. \tag{3.12}$$

Combining relations (3.11) and (3.12), we see that

$$\lim_{\substack{\sigma\to+\infty\\ \gamma_\sigma\in\Gamma}} \mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi}=0, \qquad \forall x\in\mathcal{S}_{M,\Phi}.$$

The object of approximation are the classes of Ψ-integrals of all elements belonging to the unit balls $U_{M,\Phi}$ or U^p_Φ of the spaces $\mathcal{S}_{M,\Phi}$ and S^p_Φ, respectively, under conditions that guarantee the embedding $U^p_\Phi\subset\mathcal{S}_{M,\Phi}$. The concept of Ψ-integral is introduced as follows [12]. Let $\Psi=\Psi(\mathbf{t})$ be an arbitrary function from $\Omega_\Phi(\mathscr{X})$ and let M^Ψ_Φ be the multiplier of an operator Φ generated by this function. In this case, the image x_Ψ of an element x under the mapping M^Ψ_Φ is called the Ψ-integral of an element x and is denoted by $M^\Psi_\Phi(x)=x_\Psi=\mathcal{J}^\Psi x$. In certain cases, it is convenient to call x the Ψ-derivative of x_Ψ and write $x=D^\Psi x_\Psi$. Thus, if x_Ψ is the Ψ-integral of x, then almost everywhere on $\mathbb{A}$, we have

$$\widehat{x}_\Psi=\Phi(\mathcal{J}^\Psi x)=\Psi(\mathbf{t})\widehat{x}(\mathbf{t}). \tag{3.13}$$

Let $\mathfrak{N}$ be a certain subset of $\mathscr{X}$. By $\Psi\mathfrak{N}$, we denote the set of Ψ-integrals of all $x\in\mathfrak{N}$ for which they exist. In particular, if

$$U_{M,\Phi}=\{x\in\mathcal{S}_{M,\Phi}:\|x\|_{M,\Phi}\le 1\} \qquad\text{and}\qquad U^p_\Phi=\{x\in\mathcal{S}^p_\Phi:\|x\|_{p,\Phi}\le 1\}$$

are the unit balls in certain spaces $\mathcal{S}_{M,\Phi}$ and S^p_Φ, $p>0$, then $\Psi U_{M,\Phi}$ and ΨU^p_Φ are the sets of Ψ-integrals of all $x\in U_{M,\Phi}$ and $x\in U^p_\Phi$ for which these integrals exist.

Comparing relations (3.13) and (3.9), we conclude that, as functions Ψ for which the definition of Ψ-integral is correct, one can choose any function from $\Omega_{M,\Phi}$. In this case, the inclusions $\Psi\mathcal{S}_{M,\Phi}\subset\mathcal{S}_{M,\Phi}$ and $\Psi\mathcal{S}^p_\Phi\subset\mathcal{S}_{M,\Phi}$ are valid.

4. APPROXIMATION CHARACTERISTICS

Let γ_σ, $\sigma\in(0,a)$, be a fixed set of Γ_σ. Consider the following quantities:

$$\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi}=\inf_{\lambda\in\Omega_{M,\Phi}}\|x-U_{\gamma_\sigma}(x,\lambda)\|_{M,\Phi}, \qquad x\in\mathcal{S}_{M,\Phi}, \tag{4.14}$$

$$\mathscr{E}_{\gamma_\sigma}(\Psi U_{M,\Phi})_{M,\Phi}=\sup_{x\in\Psi U_{M,\Phi}}\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi} \tag{4.15}$$

and

$$\mathscr{D}_\sigma(\Psi U_{M,\Phi})_{M,\Phi}=\inf_{\gamma_\sigma\in\Gamma_\sigma}\mathscr{E}_{\gamma_\sigma}(\Psi U_{M,\Phi})_{M,\Phi}. \tag{4.16}$$

In the case of approximating periodic functions with trigonometric polynomials, the quantity $\mathscr{E}_{\gamma_\sigma}(x)_p$ corresponds to the best approximation of the function x using polynomials of degree σ matching the set γ_σ, the quantity $\mathscr{E}_{\gamma_\sigma}(\Psi U_{M,\Phi})_{M,\Phi}$ corresponds to the exact upper bound on a given set of functions of such best approximations. The quantity $\mathscr{D}_\sigma(\Psi U_{M,\Phi})_{M,\Phi}$ resembles the trigonometric width of order σ of the set $\Psi U_{M,\Phi}$.

We also consider the following characteristics, which, in the periodic case, correspond to quantities related to the best σ-term approximation:

$$e_\sigma(x)_{M,\Phi}=\inf_{\gamma_\sigma\in\Gamma_\sigma}\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi}, \qquad x\in\mathcal{S}_{M,\Phi}. \tag{4.17}$$

Denote $\Psi U^p_{M,\Phi}=\Psi U^p_\Phi\cap\mathcal{S}_{M,\Phi}$, $0<p<\infty$, and

$$e_\sigma(\Psi U^p_{M,\Phi})_{M,\Phi}=\sup_{x\in\Psi U^p_{M,\Phi}}e_\sigma(x)_{M,\Phi}. \tag{4.18}$$

In what follows, we restrict ourselves to the case when corresponding characteristic functions $\chi_{\gamma_\sigma}(\cdot)$ belong to $\Omega_{M,\Phi}$, i.e., the operator Φ satisfies condition (A_M). In this case, according to Proposition 3.1, of major interest are quantities (4.14)–(4.18), where $\lambda(\mathbf{t}) = \chi_{\gamma_\sigma}(\mathbf{t})$. In this connection, we set

$$\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi} = \| x - U_{\gamma_\sigma}(x) \|_{M,\Phi}, \qquad x \in \mathcal{S}_{M,\Phi}, \tag{4.19}$$

$$\mathscr{E}_{\gamma_\sigma}(\Psi U_{M,\Phi})_{M,\Phi} = \sup_{x \in \Psi U_{M,\Phi}} \mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi} \tag{4.20}$$

and

$$\mathscr{D}_\sigma(\Psi U_{M,\Phi})_{M,\Phi} = \inf_{\gamma_\sigma \in \Gamma_\sigma} \mathscr{E}_{\gamma_\sigma}(\Psi U_{M,\Phi})_{M,\Phi}.$$

Similarly,

$$e_\sigma(x)_{M,\Phi} = \inf_{\gamma_\sigma \in \Gamma_\sigma} \| x - U_{\gamma_\sigma}(x) \|_{M,\Phi} \tag{4.21}$$

and

$$e_\sigma(\Psi U^p_{M,\Phi})_{M,\Phi} = \sup_{x \in \Psi U^p_{M,\Phi}} e_\sigma(x)_{M,\Phi}. \tag{4.22}$$

Theorem 4.1. *Let M be an arbitrary Orlicz function, and let $\Psi = \Psi(\mathbf{t})$ be an arbitrary function from the set $\mathscr{Y}(\mathbb{A}, \mathrm{d}\mu)$, essentially bounded on $\mathbb{A}$, i.e.,*

$$\operatorname*{ess\,sup}_{\mathbf{t} \in A} |\Psi(\mathbf{t})| < \infty \tag{4.23}$$

and in the case where the set $\mathbb{A}$ is unbounded, let

$$\lim_{|\mathbf{t}| \to +\infty} \Psi(\mathbf{t}) = 0. \tag{4.24}$$

Further, let $\mathscr{X}$ be an arbitrary linear space, and Φ be an operator that acts according to (2.3) and satisfies the condition (A_M). Then, for any $\gamma_\sigma \in \Gamma_\sigma$, $\sigma < a$, the following estimates hold:

$$\mathscr{E}_{\gamma_\sigma}(\Psi U_{M,\Phi})_{M,\Phi} \le \bar{\Psi}_{\gamma_\sigma}(0+), \tag{4.25}$$

where $\bar{\Psi}_{\gamma_\sigma}$ is the decreasing rearrangement of the function

$$\Psi_{\gamma_\sigma}(\mathbf{t}) = \begin{cases} |\Psi(\mathbf{t})|, & \mathbf{t} \in \mathbb{A} \setminus \gamma_\sigma, \\ 0, & \mathbf{t} \in \gamma_\sigma, \end{cases} \tag{4.26}$$

and

$$\mathscr{D}_\sigma(\Psi U_{M,\Phi})_{M,\Phi} \le \bar{\Psi}(\sigma+), \tag{4.27}$$

where $\bar{\Psi}$ is the decreasing rearrangement of the function $|\Psi(\mathbf{t})|$, $\mathbf{t} \in \mathbb{A}$.

If, in addition, for any $\gamma_\sigma \in \Gamma_\sigma$, $\sigma \in (0, a)$, the characteristic functions χ_{γ_σ} belong to the set $E(\Phi)$ of values of the operator Φ, and their preimages U_{γ_σ} have Ψ-integrals, then relations (4.25) and (4.27) are equalities. In this case, there exists a set γ^*_σ in Γ_σ for which the following equalities hold:

$$\mathscr{E}_{\gamma^*_\sigma}(\Psi U_{M,\Phi})_{M,\Phi} = \mathscr{D}_\sigma(\Psi U_{M,\Phi})_{M,\Phi} = \bar{\Psi}(\sigma+). \tag{4.28}$$

In particular, any measurable subset of the set $\{\mathbf{t} \in \mathbb{A} : |\Psi(\mathbf{t})| \ge \bar{\Psi}(\sigma+)\}$ that contains the set $\{\mathbf{t} \in \mathbb{A} : |\Psi(\mathbf{t})| > \bar{\Psi}(\sigma+)\}$ can be taken as γ^*_σ.

Note that conditions (4.23) and (4.24) guarantee that for the function $|\Psi(\mathbf{t})|$, its distribution function $m_{|\Psi|}(y)$,

$$m_{|\Psi|}(y) = \mathrm{mes}_\mu E_y, \qquad E_y = \{\mathbf{t} \in \mathbb{A} : |\Psi(\mathbf{t})| \ge y\}, \qquad y \ge 0,$$

takes only finite values from the interval $[0,a]$ for any $y>0$. Therefore, according to the definition of a decreasing rearrangement (see, for example, [3, Ch. 10], [4, Ch. 6], [12]), the functions $\bar{\Psi}_{\gamma_\sigma}(t)$ and $\bar{\Psi}(t)$ are always defined.

We also note that, in the case where $E(\Phi)=L_M(\mathbb{A},\,\mathrm{d}\mu)$, the operator Φ satisfies condition (A_M). Moreover, by virtue of conditions (4.23) and (4.24), the requirements that guarantee the equality in relations (4.25) and (4.27) are also satisfied.

Proof. Let's use the proof scheme of Theorem 1 of [12] and Theorem 2 of [9]. Note that the function $\Psi_{\gamma_\sigma}(\mathbf{t})$ is essentially bounded on $\mathbb{A}\backslash\gamma_\sigma$ by virtue of (4.23). Therefore, its rearrangement in decreasing order is bounded. Hence, there exists the limit

$$\bar{\Psi}_{\gamma_\sigma}(0+)=\lim_{t\to 0+}\bar{\Psi}_{\gamma_\sigma}(t)=:y_\sigma. \tag{4.29}$$

First, we prove relation (4.25). If $x\in\Psi U_{M,\Phi}$, according to (4.19), (2.4), (3.9) and (2.2), we get

$$\begin{aligned}\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi}&=\|\Phi(x-U_{\gamma_\sigma}(x))\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)}\\&=\|\widehat{x}-\chi_{\gamma_\sigma}\widehat{x}\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)}\\&=\|\,\Psi y-\chi_{\gamma_\sigma}\Psi y\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)},\end{aligned} \tag{4.30}$$

where y is some function of the unit ball $U_M(\mathbb{A},\,\mathrm{d}\mu)$ in the space $L_M(\mathbb{A},\,\mathrm{d}\mu)$. Further, we can use Theorem 2 from the paper [9], where, in fact, it is shown that

$$\sup_{y\in U_M(\mathbb{A},\,\mathrm{d}\mu)}\|\Psi y-\chi_{\gamma_\sigma}\Psi y\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)}=\bar{\Psi}_{\gamma_\sigma}(0+). \tag{4.31}$$

Combining (4.20), (4.30) and (4.31), we see that relation (4.25) is true indeed. Considering the lower bounds of both parts of (4.25) over the set Γ_σ, we get

$$\mathscr{D}_\sigma(\Psi U_{M,\Phi})_{M,\Phi}\le\inf_{\gamma_\sigma\in\Gamma_\sigma}\bar{\Psi}_{\gamma_\sigma}(0+). \tag{4.32}$$

In view of relation (4.26), we can conclude that the least value of the quantity $\bar{\Psi}_{\gamma_\sigma}(0+)$ is realized in the case where $\gamma_\sigma=\gamma_\sigma^*$, and this value is equal to $\bar{\Psi}(\sigma+)$, i.e.,

$$\inf_{\gamma_\sigma\in\Gamma_\sigma}\bar{\Psi}_{\gamma_\sigma}(0+)=\bar{\Psi}_{\gamma_\sigma^*}(0+)=\bar{\Psi}(\sigma+). \tag{4.33}$$

This proves relation (4.27).

Now assume that, for any $\gamma_\sigma\in\Gamma_\sigma$, $\sigma\in(0,a)$, the characteristic function χ_{γ_σ} belong to the set $E(\Phi)$ of values of the operator Φ, and its preimage U_{γ_σ} has Ψ-integrals, which belongs to $\mathcal{S}_{M,\Phi}$. For a given $\gamma_\sigma\in\Gamma_\sigma$, we take any number y from the interval $(0,y_\sigma)$. Assume that $e_y=\{\mathbf{t}\in\mathbb{A}\setminus\gamma_\sigma:\varphi_{\gamma_\sigma}(\mathbf{t})\ge y\}$ and $\bar{e}_y$ is any subset of the set e_y, whose μ-measure does not exceed 1, i.e., $\mathrm{mes}_\mu\bar{e}_y\le 1$. Let

$$h_y(\mathbf{t}):=\chi_{\bar{e}_y}(\mathbf{t})M^{-1}\Big((\mathrm{mes}_\mu\,\bar{e}_y)^{-1}\Big),$$

where $\chi_{\bar{e}_y}$ is a characteristics function of the set $\bar{e}_y$ and M^{-1} is the inverse function of the function M. Let also U_y be the preimage of the function h_y under the mapping of Φ, i.e., $\Phi(U_y)=h_y$, and $x_\Psi=\mathcal{J}^\Psi U_y$ be the Ψ-integral of the element U_y. By virtue of the above assumptions, all elements constructed exist, and, since

$$\int_{\mathbb{A}}M(h_y)\,\mathrm{d}\mu(\mathbf{t})\le 1,$$

we get $h_y \in U_M^+(\mathbb{A})$ and $x_\Psi \in \Psi U_{M,\Phi}$. For the element x_Ψ, relations (4.30) and (4.25) yields

$$\begin{aligned}\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi} &= \|\Phi(x_\Psi - U_{\gamma_\sigma}(x_\Psi))\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)}\\ &= \|\Psi h_y\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)}\\ &= \inf\left\{\alpha>0\ :\ \int_{\mathbb{A}\setminus\gamma_\sigma} M\big(\Psi_{\gamma_\sigma}(\mathbf{t})h_y(\mathbf{t})/\alpha\big)\,d\mu \le 1\right\}\\ &\ge y.\end{aligned}$$

Taking into account the arbitrariness of the choice of y from the interval $(0, y_\sigma)$, we can conclude that the set $\Psi U_{M,\Phi}$ contains elements x for which the values of $\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi}$ are arbitrarily close to the value of y_{γ_σ}. With regard for relation (4.29), this means that, in the case considered, relation (4.25) is, in fact, the equality. Then, according to (4.32) and (4.33), relation (4.27) is also the equality. If, in this case, if γ_σ^* is any measurable subset of the set $\{\mathbf{t}\in\mathbb{A} : |\Psi(\mathbf{t})| \ge \bar{\Psi}(\sigma+)\}$ that contains the set $\{\mathbf{t}\in\mathbb{A} : |\Psi(\mathbf{t})| > \bar{\Psi}(\sigma+)\}$, then

$$\bar{\Psi}_{\gamma_\sigma^*}(0+) = \bar{\Psi}(\sigma+).$$

Then, according to relation (4.25) (which is now the equality), we get

$$\mathscr{E}_{\gamma_\sigma^*}(\Psi U_{M,\Phi})_{M,\Phi} = \bar{\Psi}(\sigma+).$$

This yields (4.28).

□

Note that in the spaces $\mathcal{S}^p_\Phi$, the assertions, similar to Theorem 4.1, were obtained in [12, 16, 8]. In particular, in the case when $M(u) = u^p$, $p \ge 1$, the statement of Theorem 4.1 coincides with the statement of Theorem 1 in [12].

Theorem 4.2. *Let $p \in (0, +\infty)$ and M be an arbitrary Orlicz function such that the function $M(u^{1/p})$ is also Orlicz function. Let also $\Psi = \Psi(\mathbf{t})$ be an arbitrary function from the set $\mathscr{Y}(\mathbb{A}, \mathrm{d}\mu)$, essentially bounded on $\mathbb{A}$, which, in the case of an unbounded set $\mathbb{A}$ satisfies the condition (4.24). Further, let $\mathscr{X}$ be an arbitrary linear space, and Φ be an operator that acts according to (2.3) and satisfies the condition (A_M). Then for any σ, $\sigma \in (0, a)$, the following inequality holds:*

$$e_\sigma(\Psi U^p_{M,\Phi})_{M,\Phi} \le \sup_{l\in(\sigma,a]} \frac{\left(\int_0^l \bar{\Psi}^{-p}(t)dt\right)^{-\frac{1}{p}}}{M^{-1}\left(\frac{1}{l-\sigma}\right)}, \tag{4.34}$$

where M^{-1} is the inverse function of the function M and $\bar{\Psi}$ is the decreasing rearrangement of the function $|\Psi(\mathbf{t})|$, $\mathbf{t} \in \mathbb{A}$. The value of the exact upper bound in (4.34) is attained at some finite value $l = l^*$.

If, in addition, the set $E(\Phi)$ of values of the operator Φ coincides with the entire space $L_M(\mathbb{A}, \mathrm{d}\mu)$, then relation (4.34) is equality.

Proof. Let $x \in \Psi U^p_{M,\Phi}$. By virtue of (4.21), (2.5) and (3.9), we see that

$$\begin{aligned}e_\sigma(x)_{M,\Phi} &= \inf_{\gamma_\sigma\in\Gamma_\sigma} \|\Phi(x - U_{\gamma_\sigma}(x))\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)}\\ &= \inf_{\gamma_\sigma\in\Gamma_\sigma} \|\widehat{x} - \chi_{\gamma_\sigma}\widehat{x}\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)}\\ &= \inf_{\gamma_\sigma\in\Gamma_\sigma} \|\,|\Psi|y - \chi_{\gamma_\sigma}|\Psi|y\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)}\\ &= \inf_{\gamma_\sigma\in\Gamma_\sigma} \inf\left\{\alpha>0 : \int_{\mathbb{A}\setminus\gamma_\sigma} M\Big(\tfrac{|\Psi(\mathbf{t})|y(\mathbf{t})}{\alpha}\Big)d\mu(\mathbf{t}) \le 1\right\},\end{aligned}$$

where y is a certain function such that the function $|y|$ belongs to the intersection of the space $L_M(\mathbb{A},\,\mathrm{d}\mu)$ and the set $U_p^+(\mathbb{A},\mathrm{d}\mu)$ of all non-negative functions from the unit ball $U_p(\mathbb{A},\mathrm{d}\mu)$ of the space $L_p(\mathbb{A},\mathrm{d}\mu)$, i.e., $|y|\in L_M(\mathbb{A},\,\mathrm{d}\mu)\cap U_p^+(\mathbb{A},\mathrm{d}\mu)=:\mathcal{U}_{p,M}(\mathbb{A})$. Then by (4.22), we get

$$e_\sigma(\Psi U^p_{M,\Phi})_{M,\Phi}\le \sup_{h\in\mathcal{U}_{p,M}(\mathbb{A})}\ \inf_{\gamma_\sigma\in\Gamma_\sigma}\ \inf\left\{\alpha>0:\int_{\mathbb{A}\setminus\gamma_\sigma} M\Big(\tfrac{|\Psi(\mathbf{t})|y(\mathbf{t})}{\alpha}\Big)d\mu(\mathbf{t})\le 1\right\}. \tag{4.35}$$

It was shown in [9] that for any p, M and Ψ satisfying the conditions of Theorem 4.2, the following equality holds:

$$\sup_{h\in\mathcal{U}_{p,M}(\mathbb{A})}\ \inf_{\gamma_\sigma\in\Gamma_\sigma}\ \inf\left\{\alpha>0:\int\limits_{\mathbb{A}\setminus\gamma_\sigma} M\Big(\tfrac{|\Psi(\mathbf{t})|y(\mathbf{t})}{\alpha}\Big)d\mu(\mathbf{t})\le 1\right\}=\sup_{l\in(\sigma,a]}\frac{\Big(\int_0^l\bar{\Psi}^{-p}(t)dt\Big)^{-\frac{1}{p}}}{M^{-1}(\frac{1}{l-\sigma})}, \tag{4.36}$$

where M^{-1} is the inverse function of the function M and $\bar{\Psi}$ is the decreasing rearrangement of the function $|\Psi(\mathbf{t})|$, $\mathbf{t}\in\mathbb{A}$. The value of the exact upper bound in (4.34) is attained at some finite value $l=l^*$.

Combining relations (4.35) and (4.36), we obtain (4.34):

$$e_\sigma(\Psi U^p_\Phi)_{M,\Phi}\le \sup_{l\in(\sigma,a]}\frac{\Big(\int_0^l\bar{\Psi}^{-p}(t)dt\Big)^{-\frac{1}{p}}}{M^{-1}\Big(\frac{1}{l-\sigma}\Big)}=\frac{\Big(\int_0^{l^*}\bar{\Psi}^{-p}(t)dt\Big)^{-\frac{1}{p}}}{M^{-1}\Big(\frac{1}{l^*-\sigma}\Big)}.$$

Consider the function

$$y^*(\mathbf{t})=\frac{\chi_{\mathbb{E}}(\mathbf{t})}{|\Psi(\mathbf{t})|}\bigg(\int_{\mathbb{E}}|\Psi(\mathbf{u})|^{-p}d\mu(\mathbf{u})\bigg)^{-\frac{1}{p}},\qquad \mathbf{t}\in\mathbb{A},$$

where $\mathbb{E}$ is any subset of the set $\{\mathbf{t}\in\mathbb{A}:\ |\Psi(\mathbf{t})|\ge\bar{\Psi}(l^*-)\}$, which contains the set $\{\mathbf{t}\in\mathbb{A}:|\Psi(\mathbf{t})|>\bar{\Psi}(l^*-)\}$, $\mathrm{mes}_\mu\mathbb{E}=l^*$, and $\chi_{\mathbb{E}}$ is the characteristic function of the set $\mathbb{E}$.

Assume that the set $E(\Phi)$ of values of the operator Φ coincides with the space $L_M(\mathbb{A},\mathrm{d}\mu)$. Then there exists an element $x\in\mathcal{S}_{M,\Phi}$ such that $\widehat{x}(\mathbf{t})=y^*(\mathbf{t})$ almost everywhere on $\mathbb{A}$ and

$$\|x\|^p_{p,\Phi}=\|\widehat{x}\|^p_{L_p(\mathbb{A},\,\mathrm{d}\mu)}=\|y^*\|^p_{L_p(\mathbb{A},\,\mathrm{d}\mu)}=\int_{\mathbb{E}}|\Psi(\mathbf{t})|^{-p}\,\mathrm{d}\mu(\mathbf{t})\bigg(\int_{\mathbb{E}}|\Psi(\mathbf{u})|^{-p}\,\mathrm{d}\mu(\mathbf{u})\bigg)^{-1}=1,$$

therefore, $x\in U^p_\Phi$. Furthermore, there exists an element $x_\Psi\in\mathcal{S}_{M,\Phi}$ such that almost everywhere on $\mathbb{A}$

$$\widehat{x}_\Psi(\mathbf{t})=\Psi(\mathbf{t})\widehat{x}(\mathbf{t})=\Psi(\mathbf{t})y^*(\mathbf{t}).$$

Since $x\in U^p_\Phi$, we have $\widehat{x}_\Psi\in\Psi U^p_{M,\Phi}$ and the following relation holds:

$$\begin{aligned}
e_\sigma(x)_{M,\Phi}&=\inf_{\gamma_\sigma\in\Gamma_\sigma}\|\Phi(x_\Psi-U_{\gamma_\sigma}(x_\Psi))\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)}\\
&=\inf_{\gamma_\sigma\in\Gamma_\sigma}\|\widehat{x}_\Psi-\chi_{\gamma_\sigma}\widehat{x}_\Psi\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)}\\
&=\inf_{\gamma_\sigma\in\Gamma_\sigma}\|\Psi y^*-\chi_{\gamma_\sigma}\Psi y^*\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)}\\
&=\inf_{\gamma_\sigma\in\Gamma_\sigma}\inf\left\{\alpha>0:\int_{\mathbb{A}\setminus\gamma_\sigma}M\Big(\tfrac{|\Psi(\mathbf{t})|y^*(\mathbf{t})}{\alpha}\Big)\mathrm{d}\mu(\mathbf{t})\le 1\right\}\\
&=\inf\left\{\alpha>0:\ (l^*-\sigma)M\bigg(\bigg(\int_{\mathbb{E}}|\Psi(\mathbf{u})|^{-p}\,\mathrm{d}\mu(\mathbf{u})\bigg)^{-\frac{1}{p}}\Big/\alpha\bigg)\le 1\right\}.
\end{aligned}$$

For any number $h\ge 0$, the μ-measure of the set $\{\mathbf{t}\in\mathbb{E}:\ |\Psi(\mathbf{t})|\ge h\}$ is equal to the Lebesgue measure of the set $\{t\in(0,s^*):\ \bar{\Psi}(t)\ge h\}$. Therefore, from the definition of the decreasing rearrangement of a function, it follows that

$$\int_{\mathbb{E}} |\Psi(\mathbf{u})|^{-p}\,\mathrm{d}\mu(\mathbf{u}) = \int_0^{s^*} \bar{\Psi}^{-p}(t)\,\mathrm{d}t$$

and

$$e_\sigma(x)_{M,\Phi} = \inf\left\{\alpha > 0 :\ M\left(\left(\int_0^{s^*} \bar{\varphi}^{-p}(t)dt\right)^{-\frac{1}{p}}/\alpha\right) \le \tfrac{1}{s^*-\sigma}\right\} = \frac{\left(\int_0^{l^*} \bar{\Psi}^{-p}(t)dt\right)^{-\frac{1}{p}}}{M^{-1}\left(\frac{1}{l^*-\sigma}\right)}.$$

Thus, in the given case, the relation (4.34) is actually an equality.

□

Note that in the spaces $\mathcal{S}^p_\Phi$, statements similar to Theorem 4.2 were obtained in [12, 16, 8]. In particular, if $M(u) = u^q$, $q = p$, the statement of Theorem 4.2 coincides with the statement of Theorem 2 in [12]. In the case when $0 < p, q < \infty$, the results similar to Theorem 4.2 were obtained in [16]. Estimates similar to (4.36) in the spaces $L_p(\mathbb{A}, \mathrm{d}\mu)$ were proven in [17]. In the spaces $\mathcal{S}^p_\varphi$ and $\mathcal{S}^p(\mathbb{T}^d)$, statements similar to Theorem 4.1 and 4.2 were obtained in [11, 13] and [14, Chap. 11], and in the Orlicz sequence spaces l_M, such statements were proven in [10].

ACKNOWLEDGMENTS

This work is part of a project that has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement Number 873071, SOMPATY.

Conflict of interest. The author declares no potential conflict of interests.

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