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Research Article

Semi-discrete sampling operators acting on function spaces

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ABSTRACT. In this paper, we present an overview of recent advances in the study of the approximation properties of a family of semi-discrete sampling operators in several function spaces. We investigate their convergence properties in the space of continuous functions, providing an approximation result in the uniform norm and both quantitative and qualitative estimates. Here, we also establish a regularization result for functions in L^p-spaces. We then extend the study to the broader framework of Orlicz spaces, which allows the treatment of functions that are not necessarily continuous, as is often the case for real-world signals. In this setting, besides convergence, we also study the rate of approximation in terms of the φ-modulus of continuity defined by the modular functional. This unified approach yields approximation results in several particular cases, including Zygmund spaces, exponential-type spaces, and L^p-spaces. In the last setting, we are also able to achieve a sharper rate of convergence.

Keywords: Orlicz spaces, L^p-approximation, modular convergence, Durrmeyer sampling operators, Lipschitz classes, modulus of continuity.

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1. INTRODUCTION

We recall that Durrmeyer sampling operators, which are the focus of this overview, originate from the classical Bernstein polynomials, which were introduced in the context of polynomial approximation:

$$(\mathcal{B}_n f)(x) := \sum_{k=0}^{n} p_{n,k}(x)\, f\left(\frac{k}{n}\right), \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1].$$

These polynomials provided, in 1912, a constructive proof of the Weierstrass approximation theorem by algebraic polynomials in the space of continuous functions on $[0,1]$.

The Durrmeyer modification of Bernstein polynomials replaces the pointwise values $f(k/n)$ by an integral in which the same generating kernel of $\mathcal{B}_n$ appears:

$$(\mathcal{D}_n f)(x) := (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(u) f(u)\, du, \quad x \in [0,1],\ n \in \mathbb{N},$$

see, e.g., [33, 32, 20, 22, 21]. The literature on Durrmeyer-type operators is wide and includes several generalizations and variants, particularly concerning their approximation properties, convergence behavior, and asymptotic formulae [35, 36, 34, 38, 2, 1, 19].

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In this paper, we focus on the application of the Durrmeyer method to the generalized sampling series. As it is well-known, sampling-type operators were originally introduced to provide approximate versions of the celebrated Whittaker-Kotel'nikov-Shannon sampling theorem (see, e.g., [59, 43, 53]), which represents a rigorous mathematical model in signal processing. This theorem allows to reconstruct a continuous-time signal $f(t)$ at any instant t on the real line from a discrete set of sampled values $f(k/w)$, $k \in \mathbb{Z}$, $w > 0$, via an elegant interpolation formula (see, e.g., [16]).

Applying the Durrmeyer method to these sampling series leads to semi-discrete operators of the form

$$(\mathcal{D}_w^{\varphi,\psi} f)(x) := \sum_{k\in\mathbb{Z}} \varphi(wx-k)\, w \int_{\mathbb{R}} \psi(wu-k) f(u)\, du, \quad x \in \mathbb{R},\ w > 0, \tag{1.1}$$

where φ and ψ are kernel functions satisfying standard moment conditions (see [11]). These operators are commonly known as Durrmeyer sampling operators based on φ and ψ. In this formulation, the pointwise evaluation of the function is replaced by a general convolution integral with the kernel ψ. Notice that ψ generates a Fejér-type approximate identity via $\psi_w(\cdot) := w\psi(w\cdot)$, $w > 0$. In particular, the operators in (1.1) can be seen as a result of a double convolution: A continuous convolution generated by ψ (called the continuous kernel) followed by a discrete one generated by φ (called the discrete kernel). By virtue of their semi-discrete nature, from (1.1) some important particular cases arise:

- If $\psi := \chi_{[0,1]}$, the characteristic function of $[0,1]$, we obtain the Kantorovich sampling operators [28, 48, 6, 29, 30, 3, 31]:
$$(\mathcal{D}_w^{\varphi,\chi_{[0,1]}} f)(x) = \sum_{k\in\mathbb{Z}} \varphi(wx-k)\, w \int_{k/w}^{(k+1)/w} f(u)\, du =: (\mathcal{K}_w f)(x), \quad x \in \mathbb{R}.$$
- If $\psi := \delta$, the Dirac delta distribution, we recover the generalized sampling operators [14, 15, 57, 58, 42, 7]:
$$(\mathcal{D}_w^{\varphi,\delta} f)(x) = \sum_{k\in\mathbb{Z}} \varphi(wx-k) f\left(\frac{k}{w}\right) =: (\mathcal{G}_w f)(x), \quad x \in \mathbb{R}.$$

Operators in the form (1.1) were introduced by Bardaro and Mantellini to provide asymptotic expansions and Voronovskaja-type formulae for regular functions [11, 10, 12]. In particular, under suitable moment-type conditions, for $f \in C^r_{loc}(\mathbb{R})$, that is the space of r-times locally continuously differentiable functions on $\mathbb{R}$, the following local expansion holds:

$$(\mathcal{D}_w^{\varphi,\psi} f)(x) = \sum_{j=0}^{r} \frac{f^{(j)}(x)}{j!\, w^j} \sum_{\nu=0}^{j} \binom{j}{\nu} \widetilde{m}_{j-\nu}(\psi) m_\nu(\varphi) + o(w^{-r}), \quad \text{as } w \to +\infty,\ x \in \mathbb{R},$$

where $\widetilde{m}_\nu(\cdot)$ and $m_\nu(\cdot)$, $\nu \in \mathbb{N}$, denote the continuous and the discrete algebraic moments, respectively (see, e.g., Section 2). Some years later, Costarelli, Piconi and Vinti studied the convergence properties of (1.1) in more general function spaces, such as in $C(\mathbb{R})$ and in the Orlicz setting, also providing quantitative and qualitative results through suitable moduli of continuity [24, 25]; later, they also treated the multivariate case [23]. Several developments have followed, including the nonlinear version of operators in (1.1) [56], their variational properties [4], as well as extensions to weighted spaces [5] and to exponential variants of the operators [8, 18, 41]. In addition, Costarelli et al. investigated the regularization properties of (1.1) using a distributional approach, where they also obtained the distributional Fourier transform of the operators [26]. The same authors then studied higher-order approximation, proving direct estimates in terms of higher-order moduli of smoothness in L^p-spaces, along with inverse approximation results [27]. More recently, Sharma and Gupta investigated the convergence behavior of compositions of Durrmeyer sampling operators [54].

In this work, we provide an overview of some of the main results concerning the operators (1.1). We discuss convergence properties, rates of convergence, and regularization, ranging from continuous functions to the general Orlicz setting. In Section 2, we recall key tools such as the algebraic and absolute moments of the kernels and moduli of continuity in $C(\mathbb{R})$ and L^p settings. Section 3 presents the main results. In the space of continuous functions (Section 3.1), we show pointwise and uniform convergence, providing quantitative estimates in terms of the classical modulus of continuity, qualitative estimates for functions in suitable Lipschitz classes, and a regularization result showing how the operators can regularize a general function even when it is not necessarily continuous. Then, we consider the convergence properties of the operators in the more general context of Orlicz spaces (Section 3.2), which generalize L^p-spaces. Here, we provide a modular convergence theorem, from which convergence in several functional spaces, such as Zygmund or exponential-type spaces, can be deduced. Moreover, quantitative estimations are presented in terms of the modulus of continuity in Orlicz spaces, using an integral decay condition on the kernels. A special attention is devoted to the L^p-case, where sharper convergence rates can be achieved (Subsection 3.2.1).

The paper ends with final conclusions and some further developments.

2. Basic notions and preliminaries

We introduce the following notation.

For $1 \leq p \leq +\infty$, let $L^p(\mathbb{R})$ denote the usual Lebesgue space of measurable functions $f : \mathbb{R} \to \mathbb{R}$, equipped with the norm

$$\|f\|_p := \begin{cases} \left(\displaystyle\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p}, & 1 \leq p < +\infty, \\ \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)|, & p = +\infty. \end{cases}$$

Moreover, we denote by $C(\mathbb{R})$ the subspace of $L^\infty(\mathbb{R})$ consisting of all bounded and uniformly continuous functions, where $\|\cdot\|_\infty$ coincides with the usual sup-norm.

Now, let us consider two functions $\varphi, \psi \in L^1(\mathbb{R})$, with φ being bounded in a neighborhood of the origin. We define the discrete and continuous algebraic moments of φ and ψ of order ν, respectively, as follows:

$$m_\nu(\varphi, u) := \sum_{k\in\mathbb{Z}} \varphi(u-k)(k-u)^\nu, \quad \widetilde{m}_\nu(\psi) := \int_{\mathbb{R}} u^\nu \psi(u)\, du, \quad u \in \mathbb{R},\ \nu \in \mathbb{N}_0,$$

and the discrete and continuous absolute moments as

$$M_\nu(\varphi) := \sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{Z}} \big|\varphi(u-k)\big|\, |u-k|^\nu, \quad \widetilde{M}_\nu(\psi) := \int_{\mathbb{R}} |u|^\nu\, |\psi(u)|\, du, \quad \nu > 0,$$

respectively. In particular, if the following basic condition holds

$$m_0(\varphi, u) = \widetilde{m}_0(\psi) = 1, \tag{2.2}$$

we call the functions φ and ψ as discrete and continuous kernel, respectively.

We observe that if $\mu, \nu > 0$ with $\mu \leq \nu$, then $M_\nu(\varphi) < +\infty$ implies $M_\mu(\varphi) < +\infty$, and similarly $\widetilde{M}_\nu(\psi) < +\infty$ implies $\widetilde{M}_\mu(\psi) < +\infty$. In particular, if φ has compact support, then $M_\nu(\varphi) < +\infty$ for every $\nu \geq 0$.

Moreover, throughout the present paper we always assume the natural moment-condition that

$$M_0(\varphi) < +\infty.$$

A first immediate consequence of this is that the operators $\mathcal{D}_w^{\varphi,\psi}$ are well-defined for every $f \in L^\infty(\mathbb{R})$, and satisfy

$$|(\mathcal{D}_w^{\varphi,\psi} f)(x)| \leq M_0(\varphi)\, \|\psi\|_1\, \|f\|_\infty, \quad x \in \mathbb{R}. \tag{2.3}$$

Therefore, the sampling Durrmeyer operators are bounded linear operators mapping $L^\infty(\mathbb{R})$ into itself.

By a concise way, we denote by $X^p(\mathbb{R}) := L^p(\mathbb{R})$ if $1 \leq p < +\infty$ and by $X^\infty(\mathbb{R}) := C(\mathbb{R})$ if $p = +\infty$. We now recall the definition of the modulus of continuity in the space $X^p(\mathbb{R})$, $1 \leq p \leq +\infty$. (see, e.g., [17]).

Definition 2.1. *For $f \in X^p(\mathbb{R})$, the modulus of continuity is defined by*

$$\omega_p(f, \delta) := \sup_{|h| \leq \delta} \|f(\cdot + h) - f(\cdot)\|_p, \qquad \delta > 0.$$

It is well-known that for any $f \in X^p(\mathbb{R})$ there holds $\omega_p(f, \delta) \leq \omega_p(f, \delta')$ for every $0 < \delta \leq \delta'$ and the following useful inequality holds

$$\omega_p(f, \lambda\delta) \leq (1 + \lambda)\omega_p(f, \delta), \qquad \lambda, \delta > 0. \tag{2.4}$$

In this case, we can define the Lipschitz class of order $0 < \nu \leq 1$ in the space $X^p(\mathbb{R})$ for $1 \leq p \leq +\infty$ as

$$\text{Lip}(\nu, p) := \Big\{ f \in X^p(\mathbb{R}) \ : \ \|f(\cdot + h) - f(\cdot)\|_p = O(h^\nu) \text{ as } h \to 0 \Big\}, \tag{2.5}$$

or, equivalently, the space of functions in $X^p(\mathbb{R})$ for which $\omega_p(f, \delta) = O(h^\nu)$ as $h \to 0$.

3. Main results

In the following, we present some recent advances in the study of approximation properties of Durrmeyer sampling operators, both in spaces of continuous functions and in the more general framework of Orlicz spaces. We show convergence results, quantitative and qualitative estimates for the rate of convergence, as well as regularization properties. A special focus is devoted to the case of L^p-spaces.

3.1. **Approximation and regularization in $C(\mathbb{R})$.** A first objective in the space $C(\mathbb{R})$ is to study convergence in the uniform norm. Then, we establish the order of approximation through quantitative estimates based on the modulus of continuity introduced in Definition 2.1 for $p = +\infty$.

Lemma 3.1. *Let φ be a discrete kernel satisfying $M_\nu(\varphi) < +\infty$, for $\nu > 0$. Then for every $\gamma > 0$,*

$$\lim_{w \to +\infty} \sum_{|wx - k| > \gamma w} |\varphi(wx - k)| = 0,$$

uniformly with respect to $x \in \mathbb{R}$.

A proof of this lemma can be found, for instance, in [9]. Now, we are ready to claim the following pointwise and uniform convergence theorem in $C(\mathbb{R})$.

Theorem 3.1 ([24]). *Let $f \in L^\infty(\mathbb{R})$ and φ be a discrete kernel satisfying $M_\nu(\varphi) < +\infty$, for $\nu > 0$. Then, for every continuity point x of f, one has*

$$\lim_{w\to+\infty} (\mathcal{D}_w^{\varphi,\psi} f)(x) = f(x).$$

Moreover, if $f \in C(\mathbb{R})$, it follows that

$$\lim_{w\to+\infty} \|\mathcal{D}_w^{\varphi,\psi} f - f\|_\infty = 0.$$

Proof. We only prove the second statement, since the first one can be established by similar arguments. Fix $\varepsilon > 0$. By continuity of f, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$.

For a fixed $x \in \mathbb{R}$, using (2.2) we can write

$$\begin{aligned} |(\mathcal{D}_w^{\varphi,\psi} f)(x) - f(x)| &\le \sum_{k\in\mathbb{Z}} |\varphi(wx-k)|\, w \int_{\mathbb{R}} |\psi(wu-k)|\, |f(u) - f(x)|\, du \\ &= \left\{ \sum_{|wx-k|\le \frac{\delta}{2}w} + \sum_{|wx-k|> \frac{\delta}{2}w} \right\} |\varphi(wx-k)|\, w \int_{\mathbb{R}} |\psi(wu-k)|\, |f(u) - f(x)|\, du \\ &=: I_1 + I_2. \end{aligned}$$

For I_1, we note that if both $|wx - k| \le \frac{\delta}{2}w$ and $|wu - k| < \frac{\delta}{2}w$, then $|u - x| < \delta$, hence $|f(u) - f(x)| < \varepsilon$. Hence

$$I_{1,1} := \sum_{|wx-k|\le \frac{\delta}{2}w} |\varphi(wx-k)|\; w \int_{|wu-k|<\frac{\delta}{2}w} |\psi(wu-k)|\, |f(u) - f(x)|\, du < \varepsilon\, M_0(\varphi)\, \|\psi\|_1,$$

where the estimate follows from the change of variable $wu - k = y$ and the fact that $\psi \in L^1(\mathbb{R})$.

For the remaining part,

$$\begin{aligned} I_{1,2} :=& \sum_{|wx-k|\le \frac{\delta}{2}w} |\varphi(wx-k)|\; w \int_{|wu-k|>\frac{\delta}{2}w} |\psi(wu-k)|\, |f(u) - f(x)|\, du \\ \le& 2\|f\|_\infty M_0(\varphi) \int_{|y|\ge\frac{\delta}{2}w} |\psi(y)|\, dy, \end{aligned}$$

which tends to zero as $w \to \infty$, since $\psi \in L^1(\mathbb{R})$. Thus, there exists $w > 0$ sufficiently large such that $I_{1,2} \le 2\|f\|_\infty M_0(\varphi)\, \varepsilon$.

For I_2, a similar argument gives

$$I_2 \le 2\|f\|_\infty \|\psi\|_1 \sum_{|wx-k|>\frac{\delta}{2}w} |\varphi(wx-k)| < 2\|f\|_\infty \|\psi\|_1\, \varepsilon,$$

for $w > 0$ sufficiently large, as a consequence of Lemma 3.1.

Rearranging the estimates, we find

$$|(\mathcal{D}_w^{\varphi,\psi} f)(x) - f(x)| \lesssim \varepsilon,$$

for $w > 0$ sufficiently large. Since the bound is uniform in $x \in \mathbb{R}$, the claim is proved. □

After establishing convergence, we now turn to the study of the rate of convergence in the space $C(\mathbb{R})$, through a quantitative analysis using the classical modulus of continuity in this setting, and subsequently deducing the qualitative order in suitable Lipschitz classes. This analysis is summarized in the following result, whose proof is shown in [24].

Theorem 3.2 ([24]). *Assume that φ and ψ satisfy $M_1(\varphi) + \widetilde{M}_1(\psi) < +\infty$, and let $f \in C(\mathbb{R})$. Then,*

$$\|\mathcal{D}_w^{\varphi,\psi} f - f\|_\infty \leq C_{\varphi,\psi}\, \omega_\infty(f, 1/w)\,, \qquad w > 0,$$

where

$$C_{\varphi,\psi} := M_0(\varphi)\big(\widetilde{M}_0(\psi) + \widetilde{M}_1(\psi)\big) + M_1(\varphi)\widetilde{M}_0(\psi).$$

Moreover, if in addition $f \in \mathrm{Lip}(\nu, +\infty)$ with $0 < \nu \leq 1$, then

$$\|\mathcal{D}_w^{\varphi,\psi} f - f\|_\infty \leq C\, w^{-\nu}, \qquad w > 0,$$

where $C > 0$ is a suitable absolute constant depending only on φ, ψ and f.

As an example of discrete and continuous kernel satisfying the quantitative estimate, we consider the Bochner–Riesz kernel of order $\theta > 0$:

$$b_\theta(u) = \frac{2^\theta}{\sqrt{2\pi}} \frac{\Gamma(\theta+1)}{|u|^{\theta+1/2}} J_{\theta+1/2}(|u|), \qquad u \in \mathbb{R},$$

where J_λ denotes the Bessel function of order $\lambda > 0$ and Γ is the Euler gamma function (see Figure 1c). It holds that $b_\theta(u) = \mathcal{O}(|u|^{-\theta-1})$ as $|u| \to +\infty$, which implies $b_\theta \in L^1(\mathbb{R})$ and $M_\nu(b_\theta) < +\infty$ for every $0 \leq \nu < \theta$. Moreover, recalling that

$$\widehat{b_\theta}(v) = \begin{cases} (1-|v|^2)^\theta, & |v| \leq 1, \\ 0, & |v| > 1, \end{cases} \qquad v \in \mathbb{R},$$

we see that condition (2.2) is also fulfilled. This follows from the classical Poisson summation formula (see, e.g., [17]), since $\widehat{b_\theta}(k) = 1$ for all $k \in \mathbb{Z} \setminus \{0\}$ and $\widehat{b_\theta}(0) = 1$. Finally, we note that the finiteness of $M_1(b_\theta)$ holds assuming $\theta > 1$.

Corollary 3.1. *Let $f \in C(\mathbb{R})$. For any $\theta > 0$, there holds*

$$\lim_{w \to +\infty} \|\mathcal{D}_w^{b_\theta, b_\theta} f - f\|_\infty = 0.$$

In particular, if $\theta > 1$, then

$$\|\mathcal{D}_w^{b_\theta, b_\theta} f - f\|_\infty \leq C_{b_\theta}\, \omega_\infty(f, 1/w)\,, \qquad w > 0.$$

In this case, if in addition $f \in \mathrm{Lip}(\nu, +\infty)$ with $0 < \nu \leq 1$, then

$$\|\mathcal{D}_w^{b_\theta, b_\theta} f - f\|_\infty \leq C\, w^{-\nu}, \qquad w > 0,$$

where $C > 0$ is a suitable absolute constant depending only on b_θ and f.

Once convergence and the corresponding rate have been established, we now turn our attention to a regularization property. In fact, the following result shows that the smoothness of the operators $\mathcal{D}_w^{\varphi,\psi}$ is strongly influenced by the regularity of the discrete kernel φ. In particular, if φ is continuous and $f \in L^p(\mathbb{R})$ with $1 \leq p \leq +\infty$, then $\mathcal{D}_w^{\varphi,\psi} f$ is continuous for every $w > 0$. Hence, even when starting from a function f that is not continuous, the resulting Durrmeyer sampling operator yields a continuous function, meaning that these operators act as regularizers.

Theorem 3.3 ([26]). *Let $f \in L^p(\mathbb{R})$, with $1 \leq p \leq +\infty$. If φ is continuous (resp. uniformly continuous) and bounded, then $\mathcal{D}_w^{\varphi,\psi} f$ is continuous (resp. uniformly continuous) and bounded for every $w > 0$.*

Proof. Let $1 \le p < +\infty$ and fix $w > 0$. For $m \in \mathbb{N}^+$, define the truncated operators

$$d_w^m(x) := \sum_{|k|\le m} \Big(w \int_{\mathbb{R}} \psi(wu-k) f(u)\, du \Big) \varphi(wx-k), \quad x \in \mathbb{R}.$$

Then

$$\big|(\mathcal{D}_w^{\varphi,\psi} f)(x) - d_w^m(x)\big| \le \sum_{|k|>m} \Big| w \int_{\mathbb{R}} \psi(wu-k) f(u)\, du \Big| \, |\varphi(wx-k)|.$$

Fix $x \in \mathbb{R}$. By applying Jensen inequality, Fubini-Tonelli theorem, and using the boundedness of φ, one obtains

$$\big|(\mathcal{D}_w^{\varphi,\psi} f)(x) - d_w^m(x)\big| \le C\, w^{1/p} \|f\|_p \left(\sup_{u\in\mathbb{R}} \sum_{|k|>m} |\psi(wu-k)| \right)^{1/p},$$

for a suitable constant C depending only on φ and ψ. As $m \to +\infty$, the last term tends to zero, since $\sum_{k\in\mathbb{Z}} |\psi(wu-k)|$ converges uniformly in u by the finiteness of $M_0(\psi)$. Hence, d_w^m converges uniformly to $\mathcal{D}_w^{\varphi,\psi} f$ on $\mathbb{R}$ as $m \to +\infty$. Being the uniform limit of continuous functions by virtue of the continuity of φ, the operator $\mathcal{D}_w^{\varphi,\psi} f$ is itself continuous.

Moreover, by similar estimates one shows that there is a suitable constant $C' > 0$ depending only on φ and ψ such that

$$|(\mathcal{D}_w^{\varphi,\psi} f)(x)| \le C' w^{1/p} \|f\|_p, \quad x \in \mathbb{R},$$

so that $\mathcal{D}_w^{\varphi,\psi} f$ is also bounded. Therefore, $\mathcal{D}_w^{\varphi,\psi} f$ is continuous and bounded on $\mathbb{R}$ for $1 \le p < +\infty$.

For the case $p = +\infty$, one proceeds analogously by considering the symmetric kernel $\check{\psi}_w(\cdot) := w\psi(-w\cdot)$, leading to

$$\big|(\mathcal{D}_w^{\varphi,\psi} f)(x) - d_w^m(x)\big| \le \|\check{\psi}_w * f\|_\infty \sup_{x\in\mathbb{R}} \sum_{|k|>m} |\varphi(wx-k)|,$$

where the remainder again vanishes as $m \to +\infty$, since $M_0(\varphi) < +\infty$. Here, by $*$ we denote the usual convolution product. Moreover, by virtue of (2.3), we conclude that $\mathcal{D}_w^{\varphi,\psi} f$ is also continuous and bounded in this case.

To prove uniform continuity, observe that if $\varphi \in C(\mathbb{R})$, then each truncated operator d_w^m is uniformly continuous on $\mathbb{R}$, being a finite sum of uniformly continuous functions. Fix $\varepsilon > 0$. Since $d_w^m \to \mathcal{D}_w^{\varphi,\psi} f$ uniformly as $m \to +\infty$, there exists $m \in \mathbb{N}^+$ such that

$$\big|\mathcal{D}_w^{\varphi,\psi} f(x) - d_w^m(x)\big| < \tfrac{\varepsilon}{3}, \quad x \in \mathbb{R}.$$

Moreover, by uniform continuity of d_w^m, there exists $\delta > 0$ such that $|x-y| < \delta$ implies $|d_w^m(x) - d_w^m(y)| < \frac{\varepsilon}{3}$. Hence, for $|x-y| < \delta$,

$$\big|\mathcal{D}_w^{\varphi,\psi} f(x) - \mathcal{D}_w^{\varphi,\psi} f(y)\big| \le \big|\mathcal{D}_w^{\varphi,\psi} f(x) - d_w^m(x)\big| + \big|d_w^m(x) - d_w^m(y)\big| + \big|d_w^m(y) - \mathcal{D}_w^{\varphi,\psi} f(y)\big| < \varepsilon,$$

for m sufficiently large. Therefore, $\mathcal{D}_w^{\varphi,\psi} f$ is uniformly continuous. This completes the proof. □

For further examples and additional results on regularization in the case of continuous kernels, the reader may see [26].

3.2. **Approximation in $L^\eta(\mathbb{R})$.** Orlicz spaces originated in the 1930s as a natural extension of Lebesgue spaces [46, 47]. They are a remarkable example of modular spaces, introduced to generalize the notion of normed linear spaces. In what follows, we provide a brief overview of the fundamental notions related to Orlicz spaces [45, 44, 50, 51, 13, 37].

Let (Ω, Σ, μ) be a measure space with a σ-finite, complete measure μ. Let $\eta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a φ-function, i.e., a non-decreasing continuous function with $\eta(0) = 0$, $\eta(u) > 0$ for $u > 0$ and $\lim_{u\to+\infty} \eta(u) = +\infty$. Denoting by $L^0(\Omega, \Sigma, \mu) = L^0(\Omega)$ the space of all real-valued, Σ-measurable functions on Ω, finite μ-almost everywhere, with equality μ-a.e., the functional $I^\eta : L^0(\Omega) \to \mathbb{R}_0^+$ defined as

$$I^\eta[f] := \int_\Omega \eta(|f(x)|)\, d\mu(x), \quad f \in L^0(\Omega),$$

is proved to be a modular (see, e.g., [13, 45]). Then, the Orlicz space generated by η is given by

$$L^\eta(\Omega) := \left\{ f \in L^0(\Omega) : I^\eta[\lambda f] < +\infty, \text{ for some } \lambda > 0 \right\}.$$

In cases where the modular I^η is convex (which holds when η itself is convex), we can define the so-called Luxemburg norm as follows

$$\|f\|_\eta := \inf \left\{ u > 0 : I^\eta \left[\frac{f}{u} \right] \leq 1 \right\}.$$

In this case, the Orlicz space $L^\eta(\Omega)$ endowed with the norm $\|\cdot\|_\eta$ is a normed linear space (see Theorem 1.1 (b) of [13]). It is also a Banach space (see Theorem 3.3.7 (b) of [37]).

A φ-function η is said to satisfy the Δ_2-condition (in symbol, $\eta \in \Delta_2$) if there exists a constant $C_\eta > 0$ such that

$$\eta(u) \leq C_\eta \eta(u) \quad \text{for all } u \geq 0.$$

Norm convergence can be characterized in terms of the modular, as the following lemma shows. This is often useful, since the exact value of the norm may be difficult to compute due to its definition as an infimum.

Lemma 3.2. *Let η be a convex φ-function. Then $\|f_k\|_\eta \to 0$ as $k \to +\infty$ if and only if*

$$\lim_{k\to+\infty} I^\eta[\lambda f_k] = 0 \quad \textit{for all } \lambda > 0.$$

A weaker and natural notion of convergence in $L^\varphi(\Omega)$ is the modular convergence. More precisely, a net of functions $(f_k)_{k>0} \subset L^\eta(\Omega)$ is said to converge modularly to a function $f \in L^\eta(\Omega)$, denoted by $f_k \xrightarrow{I^\eta} f$, if

$$\lim_{k\to+\infty} I^\eta[\lambda(f_k - f)] = 0, \quad \text{for some } \lambda > 0. \tag{3.6}$$

In some cases, modular convergence and norm convergence coincide.

Lemma 3.3. *Let η be a convex φ-function such that $\eta \in \Delta_2$. Then modular convergence and norm convergence are equivalent.*

It is also possible to define the notion of modular continuity in Orlicz spaces (see, e.g., [40]).

Definition 3.2. *Let η and ζ be a pair of φ-functions. A linear operator $T : L^\eta(\Omega) \to L^\zeta(\Omega)$ is said to be modularly continuous if $f_k \in L^\eta(\Omega)$, $f_k \xrightarrow{I^\eta} f$ as $k \to +\infty$, implies $T(f_k) \xrightarrow{I^\zeta} T(f)$ as $k \to +\infty$.*

Orlicz spaces include a wide range of functional spaces with several applications in various areas of pure and applied functional analysis, such as Fourier analysis, interpolation theory, partial differential equations, and distribution theory. A classical example of a φ-function satisfying the Δ_2-condition is $\varphi(u) = u^p$, with $u \geq 0$ and $1 \leq p < +\infty$, in which case $L^\varphi(\Omega) = L^p(\Omega)$, corresponding to the classical Lebesgue spaces. Here, $\|\cdot\|_\varphi = \|\cdot\|_p$.

Other examples of Orlicz spaces are, for instance, the interpolation spaces $L^\alpha \log^\beta L(\Omega)$ (also known as Zygmund spaces [60, 55]), which are useful from the applicative point of view. These spaces are generated by the φ-function

$$\eta_{\alpha,\beta}(u) := u^\alpha \log^\beta(u+e), \quad \alpha \geq 1,\ \beta > 0,\ u \geq 0.$$

The corresponding modular functional is defined as

$$I^{\eta_{\alpha,\beta}}[f] := \int_\Omega |f(u)|^\alpha \log^\beta(e + |f(u)|)\, du, \quad f \in M(\Omega).$$

It is well known that $\eta_{\alpha,\beta} \in \Delta_2$.

On the other hand, for the so-called exponential type spaces [52, 39], i.e., the Orlicz spaces generated by

$$\eta_\gamma(u) = e^{u^\gamma} - 1, \quad \gamma > 0,\ u \geq 0,$$

the Δ_2-condition is not satisfied, and consequently, modular and norm convergence are not equivalent. The related modular functional is given by

$$I^{\eta_\gamma}[f] := \int_\Omega \left(e^{|f(u)|^\gamma} - 1\right) du, \quad f \in M(\Omega).$$

We are now ready to present some approximation results for Durrmeyer sampling operators in the general framework of the Orlicz space $L^\eta(\Omega)$, with $\Omega = \mathbb{R}$, μ the Lebesgue measure, and Σ the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$. This provides a unified treatment, in this general context, for the study of the approximation properties of the class of operators considered here.

We begin with the following theorem, which shows that the Durrmeyer sampling operators are well defined on $L^\eta(\mathbb{R})$. Moreover, a modular convergence result holds in general; if the involved φ-function additionally satisfies the Δ_2-condition, a norm convergence result is also valid.

Theorem 3.4 ([24]). *Let η be a convex φ-function and $f \in L^\eta(\mathbb{R})$ be fixed. Moreover, let $M_0(\psi) < +\infty$. Then the following statements hold:*

(1) There exists $\lambda > 0$ such that

$$I^\eta[\lambda\, \mathcal{D}_w^{\varphi,\psi} f] \;\leq\; \frac{M_0(\psi)\|\varphi\|_1}{M_0(\varphi)\|\psi\|_1}\, I^\eta[\lambda\, M_0(\varphi)\, \|\psi\|_1\, f] < +\infty, \qquad w > 0.$$

In particular, the operators $\mathcal{D}_w^{\varphi,\psi}$ are well defined and belong to $L^\eta(\mathbb{R})$ for every $w > 0$.

(2) There exists $\lambda > 0$ such that

$$\lim_{w \to +\infty} I^\eta\big[\lambda\big(\mathcal{D}_w^{\varphi,\psi} f - f\big)\big] = 0.$$

(3) If $\eta \in \Delta_2$ then

$$\lim_{w \to +\infty} \|\mathcal{D}_w^{\varphi,\psi} f - f\|_\eta = 0.$$

For a proof, carried out through a unifying but technical direct modular estimate, the reader may refer to [24]. A key point in proving convergence is that the translated function $\Delta f_h = f(\cdot + h) - f(\cdot)$ converges modularly to 0, that is, $\Delta f_h \xrightarrow{I^\eta} 0$ as $h \to 0$.

As an immediate consequence of Theorem 3.4, the operators $\mathcal{D}_w^{\varphi,\psi}$ are also modularly continuous on $L^\eta(\mathbb{R})$ for every fixed $w > 0$, according to Definition 3.2. Indeed, since there exists $\lambda^* > 0$ with $I_\eta[\lambda^*(f - f_k)] \to 0$ as $k \to +\infty$, choosing $\lambda > 0$ so that $\lambda M_0(\varphi)\|\psi\|_1 \leq \lambda^*$, we have

$$I^\eta[\lambda(\mathcal{D}_w^{\varphi,\psi} f - \mathcal{D}_w^{\varphi,\psi} f_k)] \leq \frac{M_0(\psi)\|\varphi\|_1}{M_0(\varphi)\|\psi\|_1} I^\eta[\lambda^*(f - f_k)] \to 0, \quad k \to +\infty.$$

As example of convergence of the operators in the Orlicz setting, we consider the Fejér kernel, defined as

$$\mathrm{F}(u) = \frac{1}{2}\mathrm{sinc}^2\left(\frac{u}{2}\right), \quad u \in \mathbb{R},$$

where $\mathrm{sinc}(u) = \sin(\pi u)/\pi u$ if $u \in \mathbb{R} \setminus \{0\}$ and 1 if $u = 0$ (see Figure 1a). The kernel is bounded, non-negative on $\mathbb{R}$, and belongs to $L^1(\mathbb{R})$. Moreover, condition (2.2) is satisfied, being $\widehat{\mathrm{F}}(2k\pi) = 0$ for $k \in \mathbb{Z} \setminus \{0\}$ and $\widehat{\mathrm{F}}(0) = 1$, where

$$\widehat{\mathrm{F}}(v) = \begin{cases} 1 - |\frac{v}{\pi}|, & |v| \leq \pi, \\ 0, & |v| > \pi, \end{cases} \quad v \in \mathbb{R}.$$

In particular, $\|\mathrm{F}\|_1 = M_0(\mathrm{F}) = 1$. In this case, we obtain the following.

Corollary 3.2. *Let η be a convex φ-function and $f \in L^\eta(\mathbb{R})$ be fixed. Thus, there exists $\lambda > 0$ such that*

$$I^\eta[\lambda\, \mathcal{D}_w^{\mathrm{F},\mathrm{F}} f] \leq I^\eta[\lambda\, f], \qquad w > 0.$$

In particular, there exists $\lambda > 0$ such that

$$\lim_{w \to +\infty} I^\eta[\lambda(\mathcal{D}_w^{\mathrm{F},\mathrm{F}} f - f)] = 0.$$

Now, in order to study the rate of convergence, we need the following tool.

Definition 3.3. *Let η be a φ-function. The map $\omega_\eta : L^0(\mathbb{R}) \times \mathbb{R}^+ \to [0, +\infty]$ defined by*

$$\omega_\eta(f, \delta) := \sup_{|h| \leq \delta} I^\eta[f(h + \cdot) - f(\cdot)],$$

for $f \in L^0(\mathbb{R})$, is called the η-modulus of continuity.

Lemma 3.4. *Let η be a φ-function. Then for every function $f \in L^0(\mathbb{R})$ there exists $\lambda > 0$ such that*

$$\omega_\eta(\lambda f, \delta) \to 0 \quad \text{as } \delta \to 0^+.$$

For a proof, see Theorem 2.4 of [13].

As in the classical case, we now introduce the notion of Orlicz-Lipschitz classes defined in terms of the modular functional I^η. Let $\mathcal{T}$ be the class of measurable functions $\tau : \mathbb{R} \to [0, +\infty]$ such that $\tau(t) > 0$ for all $t \neq 0$.

Definition 3.4. *For a given $\tau \in \mathcal{T}$, we define the Orlicz-Lipschitz class as*

$$\mathrm{Lip}(\tau, \eta) := \left\{ f \in L^\eta(\mathbb{R}) : \exists \lambda > 0 \text{ with } I^\eta\left[\lambda\left(f(\cdot + h) - f(\cdot)\right)\right] = \mathcal{O}(\tau(h)),\ h \to 0 \right\},$$

where, for any two functions $f, g \in L^\eta(\mathbb{R})$,

$$f(t) = \mathcal{O}(g(t)), \quad t \to 0$$

means that there exist a constant $C > 0$ and some $\delta > 0$ such that $|f(t)| \leq C|g(t)|$ for $|t| \leq \delta$.

To obtain quantitative estimates in terms of the η-modulus of continuity, we require the following further condition on the kernels.

For any $0 < \alpha < 1$, we say that a function $\xi : \mathbb{R} \to \mathbb{R}$ satisfies the integral decay condition (α) if

$$(\alpha) \qquad w \int_{|u|>1/w^{\alpha}} |\xi(wu)|\, du \leq K w^{-\mu}, \quad \text{as } w \to +\infty,$$

for suitable constants $K, \mu > 0$ depending on α and ξ.

We are now ready to state the main results about the rate of convergence of Durrmeyer sampling operators in Orlicz spaces. From the quantitative estimate below, we directly deduce the qualitative order of approximation, assuming f belongs to a suitable Orlicz-Lipschitz class. In particular, we use the Lipschitz classes from Definition 3.4, with $\tau(h) = h^{\nu}$, $0 < \nu \leq 1$. In this case, we denote the resulting Lipschitz class simply by $\mathrm{Lip}(\nu, \eta)$.

Theorem 3.5 ([25]). *Let η be a convex φ-function and let $f \in L^{\eta}(\mathbb{R})$. Moreover, let $M_0(\psi) < +\infty$. Suppose $0 < \alpha < 1$ and that both kernels φ and ψ satisfy condition (α).*

Then there exist constants $K > 0$ and $\mu > 0$, depending on α, φ, ψ, such that

$$I^{\eta}\big[\lambda\big(\mathcal{D}_w^{\varphi,\psi} f - f\big)\big] \leq \left(\frac{M_0(\psi)\|\varphi\|_1 + M_0(\varphi)\|\psi\|_1}{2M_0(\varphi)\|\psi\|_1}\right) \omega_{\eta}\big(2\lambda M_0(\varphi)\|\psi\|_1 f, w^{-\alpha}\big) + K\left(\frac{M_0(\varphi) + M_0(\psi)}{2M_0(\varphi)\|\psi\|_1}\right) I^{\eta}[4\lambda M_0(\varphi)\|\psi\|_1 f]\, w^{-\mu},$$

for $\lambda > 0$ and every sufficiently large $w > 0$. In particular, if $\lambda > 0$ is small enough, the inequality implies the modular convergence of the Durrmeyer sampling operators $\mathcal{D}_w^{\varphi,\psi} f$ to f.

Moreover, if $f \in \mathrm{Lip}(\nu, \eta)$ with $0 < \nu \leq 1$, then there exist constants $C, \lambda > 0$ such that

$$I^{\eta}\big[\lambda\big(\mathcal{D}_w^{\varphi,\psi} f - f\big)\big] \leq C w^{-\rho},$$

for all sufficiently large $w > 0$, where $\rho := \min\{\alpha\nu, \mu\}$.

Proof. Fix $\lambda > 0$. By convexity of η, we can write

$$I^{\eta}\big[\lambda\big(D_w^{\varphi,\psi} f - f\big)\big] \leq \tfrac{1}{2}(I_1 + I_2),$$

where

$$I_1 := \int_{\mathbb{R}} \eta\left(2\lambda\left|(D_w^{\varphi,\psi} f)(x) - \sum_{k\in\mathbb{Z}} \varphi(wx-k) w \int_{\mathbb{R}} \psi(wu-k) f\big(u + x - \tfrac{k}{w}\big)\, du\right|\right) dx,$$

and

$$I_2 := \int_{\mathbb{R}} \eta\left(2\lambda\left|\sum_{k\in\mathbb{Z}} \varphi(wx-k) w \int_{\mathbb{R}} \psi(wu-k) f\big(u + x - \tfrac{k}{w}\big)\, du - f(x)\right|\right) dx.$$

For I_1, using Jensen inequality twice, the change of variable $y = x - k/w$ and Fubini-Tonelli theorem, we obtain

$$I_1 \leq \frac{M_0(\psi)}{M_0(\varphi)\|\psi\|_1} \int_{\mathbb{R}} w|\varphi(wy)| \left(\int_{\mathbb{R}} \eta(2\lambda M_0(\varphi)\|\psi\|_1 \, |f(u+y) - f(u)|)\, du\right) dy.$$

Let $0 < \alpha < 1$ of condition (α) be fixed. Splitting the integral into the intervals $|y| \leq 1/w^{\alpha}$ and $|y| > 1/w^{\alpha}$, we obtain

$$I_1 \leq \frac{M_0(\psi)\|\varphi\|_1}{M_0(\varphi)\|\psi\|_1} \omega_{\eta}\big(2\lambda M_0(\varphi)\|\psi\|_1 f, w^{-\alpha}\big) + \frac{M_0(\psi)}{M_0(\varphi)\|\psi\|_1} I^{\eta}[4\lambda M_0(\varphi)\|\psi\|_1 f]\, K_{\varphi} w^{-\mu_{\varphi}},$$

thanks to the convexity of η, the translation invariance of

$$I^{\eta}[4\lambda M_0(\varphi)\|\psi\|_1 f(\cdot)] = I^{\eta}[4\lambda M_0(\varphi)\|\psi\|_1 f(\cdot + y)]$$

for every $y \in \mathbb{R}$ and condition (α) for φ.

Let us now estimate I_2. After the change of variable $t = u - \frac{k}{w}$, we obtain

$$I_2 = \int_{\mathbb{R}} \eta\left(2\lambda\left|\sum_{k\in\mathbb{Z}} \varphi(wx-k)\right|\left|\int_{\mathbb{R}} \psi(wt)\left[f(t+x)-f(x)\right]dt\right|\right) dx.$$

Applying Jensen inequality and using (2.2), this gives

$$I_2 \leq \frac{w}{\|\psi\|_1}\int_{\mathbb{R}} |\psi(wt)|\, I^{\eta}[2\lambda M_0(\varphi)\|\psi\|_1\, |f(t+\cdot)-f(\cdot)|]\, dt.$$

Splitting the integral into $|t| \leq 1/w^{\alpha}$ and $|t| > 1/w^{\alpha}$, we get

$$I_2 \leq \omega_{\eta}\big(2\lambda M_0(\varphi)\|\psi\|_1 f, w^{-\alpha}\big) + \frac{1}{\|\psi\|_1} I^{\eta}[4\lambda M_0(\varphi)\|\psi\|_1 f]\, K_{\psi} w^{-\mu_{\psi}},$$

since ψ also satisfies (α).

Combining the above two estimates, and setting

$$K := \max\{K_{\varphi}, K_{\psi}\}, \qquad \mu := \min\{\mu_{\varphi}, \mu_{\psi}\},$$

we obtain the desired inequality.

The second part of the statement, namely the qualitative estimate, directly follows from the above quantitative bound together with Definition 3.4. This concludes the proof. □

We highlight that condition (α) is satisfied by several examples of kernels, not necessarily only those with compact support, for which it becomes trivially verified. For instance, in the case of the Bochner Riesz kernel b_{θ}, for any fixed $0 < \alpha < 1$, one can take $K = \widetilde{M}_{\nu}(b_{\theta})$ and $\mu = \nu(1-\alpha)$ for every $0 \leq \nu < \theta$. This follows from the fact that $b_{\theta}(u) = O(|u|^{-\theta-1})$ as $|u| \to +\infty$. Similarly, for the Fejér kernel F, one can show that $K = \widetilde{M}_{\nu}(\mathrm{F})$ and $\mu = \nu(1-\alpha)$ for every $0 \leq \nu < 1$.

3.2.1. *Sharp analysis in L^p-spaces.* Herein, we focus on the particular case of L^p-spaces, where we are able to get a sharper order of approximation with respect to the one obtained in the Orlicz case.

As a consequence of Theorem 3.4, we obtain the following result regarding the well-definedness and convergence properties of the Durrmeyer sampling operators in L^p-spaces.

Corollary 3.3 ([24]). *Let $M_0(\varphi) < +\infty$. The following assertions hold:*

(1) *For every $f \in L^p(\mathbb{R})$, with $1 \leq p < +\infty$, we have*

$$\|\mathcal{D}_w^{\varphi,\psi} f\|_p \leq M_0(\psi)^{\frac{1}{p}} M_0(\varphi)^{\frac{p-1}{p}} \|\varphi\|_1^{\frac{1}{p}} \|\psi\|_1^{\frac{p-1}{p}} \|f\|_p, \qquad w > 0.$$

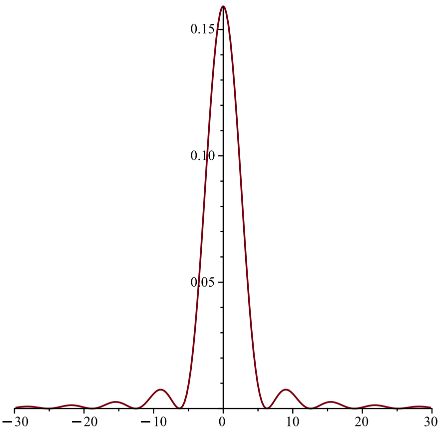
In particular, $\mathcal{D}_w^{\varphi,\psi} f$ is well defined in $L^p(\mathbb{R})$ and belongs to $L^p(\mathbb{R})$ whenever $f \in L^p(\mathbb{R})$.

(2) *For every $f \in L^p(\mathbb{R})$, with $1 \leq p < +\infty$, we have*

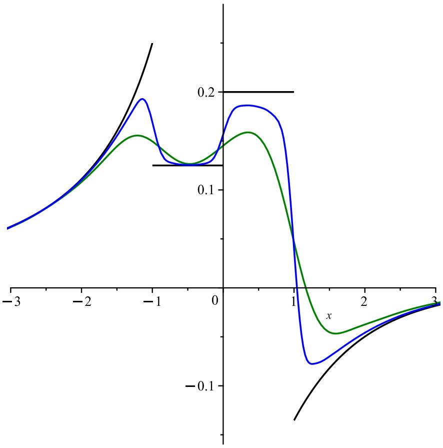
$$\lim_{w\to+\infty} \|\mathcal{D}_w^{\varphi,\psi} f - f\|_p = 0.$$

For a graphical representation, the reader can see Figures 1b and 1d.

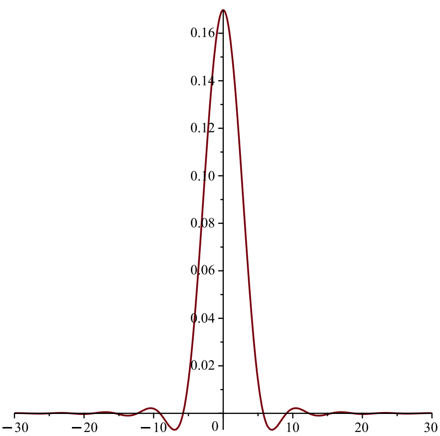
Regarding the rate of convergence, in the case of L^p-spaces we are able to achieve a sharper order compared with the one stated in the general Orlicz-space setting (Theorem 3.5).



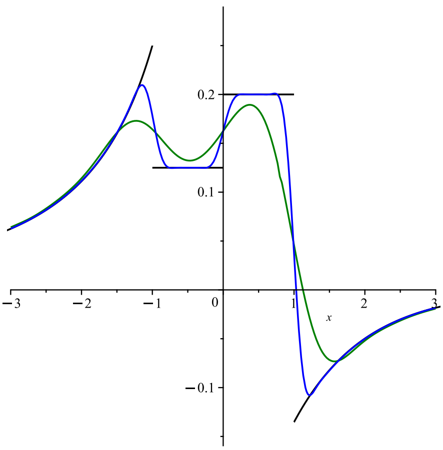
(A) Graph of the Fejér kernel.



(B) Approximation of a discontinuous function (black line) using Durrmeyer sampling operators generated by the Fejér kernel $\mathcal{D}_w^{\mathrm{F},\mathrm{F}}$ for $w = 10$ (green line) and $w = 30$ (blue line).



(C) Graph of the Bochner-Riesz kernel of order $\theta = 2$.



(D) Approximation of the same discontinuous function (black line) using Durrmeyer sampling operators generated by the Bochner-Riesz kernel (of order $\theta = 2$) $\mathcal{D}_w^{b_2,b_2}$ for $w = 10$ (green line) and $w = 30$ (blue line).

FIGURE 1. Comparison between Fejér and Bochner-Riesz kernels and their corresponding Durrmeyer approximations of a discontinuous function in $L^1(\mathbb{R})$.

Corollary 3.4 ([25]). *Let* $1 \leq p < +\infty$ *and let* φ, ψ *be such that*

$$\widetilde{M}_p(\varphi) + \widetilde{M}_p(\psi) < +\infty.$$

Then, for every $f \in L^p(\mathbb{R})$, we have the quantitative estimate

$$\|\mathcal{D}_w^{\varphi,\psi} f - f\|_p \leq M_0(\varphi)\,(2\|\psi\|_1)^{\frac{p-1}{p}} \times \left[\left(M_0(\psi)\|\varphi\|_1 + \frac{M_p(\varphi)}{M_0(\varphi)} \right)^{\frac{1}{p}} + (\|\psi\|_1 + M_p(\psi))^{\frac{1}{p}} \right] \omega_p\big(f, w^{-1}\big)\,,$$

for all sufficiently large $w > 0$.

Moreover, if $f \in \mathrm{Lip}(\nu, p)$ with $0 < \nu \leq 1$, then there exists a constant $C > 0$ such that

$$\|\mathcal{D}_w^{\varphi,\psi} f - f\|_p \leq M_0(\varphi)\,(2\|\psi\|_1)^{\frac{p-1}{p}} \left[\left(M_0(\psi)\|\varphi\|_1 + \frac{M_p(\varphi)}{M_0(\varphi)} \right)^{\frac{1}{p}} + (\|\psi\|_1 + M_p(\psi))^{\frac{1}{p}} \right] C w^{-\nu},$$

for every sufficiently large $w > 0$.

For a detailed proof, see [25]. The sharper quantitative estimates in Corollary 3.4 are obtained using a more direct approach than in the general setting of Theorem 3.5. This is possible thanks to the specific inequality, recalled in (2.4), satisfied by the L^p-modulus of smoothness ω_p, which does not hold in general in Orlicz spaces. As a result, the convergence rates in the Lipschitz classes $\mathrm{Lip}(\nu, p)$, introduced in (2.5) for $1 \leq p < +\infty$, are improved.

Conclusions and Future Developments

In this work, we have provided a general overview of some of the main approximation results for semi-discrete Durrmeyer-type operators, considering the problem in different functional settings. In the space of continuous functions, we established a theorem of pointwise and uniform convergence, supported by both quantitative and qualitative estimates, together with a regularization result in which the discrete kernel plays a crucial role. We then moved to the framework of Orlicz spaces, where we proved convergence results and obtained quantitative estimates through suitable moduli of continuity defined in terms of the modular. A particular focus has been devoted to the case of L^p-spaces, where a sharper order of approximation can be achieved.

The study of convergence, approximation order, and regularization by a distributional approach has opened new directions of research. In particular, we investigated quantitative estimates based on higher-order moduli of smoothness, which allow us to establish conditions ensuring higher-order convergence in L^p-spaces, of order $r > 1$ for some integer r. We also considered the delicate problem of inverse approximation, that is, deducing regularity properties of a function from the knowledge of the convergence rate of Durrmeyer operators in L^p-spaces. By combining direct and inverse results, we obtained a full characterization of the well-known generalized Lipschitz classes in the L^p-setting [27].

Finally, regarding the Orlicz framework, our research is moving towards the study of convergence of operators in Sobolev-Orlicz spaces, which extend Sobolev spaces in the same way that Orlicz spaces extend L^p-spaces [49]. In this context, we are investigating simultaneous approximation results for the derivatives of the operators to the derivatives of the functions, which represent a natural line of further developments.

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