



Research Article

# Wavelet-based approximation operators: applications to bivariate functions and digital image processing

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**ABSTRACT.** This work is a continuation of the author's very recent studies on the newly introduced wavelet based approximation operators, especially Bernstein operators [14]. The main goal of the present study is to construct and investigate bivariate case of these operators. In accordance with this purpose, we introduce two dimensional wavelet type Bernstein operators via wavelets and examine some characteristic properties together with their approximation results. Moreover, we give some application to bivariate functions and digital image processing.

**Keywords:** Bivariate Bernstein operators, wavelets, compactly supported Daubechies wavelets, image processing.

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## 1. INTRODUCTION

The efficient extraction of information from limited data has long been a central theme in mathematics and signal analysis. A classical illustrative example is provided by a well-known riddle from the old Turkish tradition. Let us start by telling a very beautiful, funny and clever story from the old times, that is, the time of the sultans.

A sultan commands each of his ten goldsmiths to forge ten golden balls, each expected to weigh exactly one kilogram. However, it is revealed that one of the goldsmiths has used inferior material, producing balls that are lighter by exactly 0.1 kilograms each. The challenge is to determine, using only a single measurement on a precision scale, which goldsmith is dishonest.

The solution proceeds by a weighted sampling strategy: one ball is taken from the first goldsmith, two from the second, and so on, until ten are taken from the tenth. If all the balls were genuine, the total mass should be exactly 55 kilograms. Any deviation from this expected value identifies the fraudulent goldsmith, since the deficit in weight is proportional to the number of balls sampled from the guilty party. For example, an observed deficit of 0.3 kilograms indicates the third goldsmith as the source of the anomaly. Thus, with a single carefully constructed measurement, maximal information is obtained.

This problem serves as a compelling analogy for wavelet theory. Much like the sultan's strategy, wavelet analysis is designed to uncover hidden structure in signals through efficient representation. A wavelet transform decomposes a signal into a hierarchy of subspaces at multiple resolutions, enabling both localization and sparsity in representation. This multiresolution framework allows significant features—such as singularities, discontinuities, or oscillatory

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components—to be identified using relatively few coefficients, in contrast to traditional Fourier methods that distribute information across a global basis.

Just as we distinguish the fake goldsmith by a clever sampling strategy, wavelets distinguish hidden patterns in signals by analyzing them at multiple resolutions using very few coefficients.

### Sultan's Problem

*One measurement identifies anomaly*  
*Strategy uses weighted sample*  
*Find anomaly*  
*Efficient logic*

### Wavelet Analysis

*One transform reveals key features*  
*Filter banks and scaling functions*  
*Detect feature*  
*Sparse encoding via MRA*

The underlying principle common to both is that of information economy: the ability to reveal essential structure with minimal data. In modern applications, this philosophy is embodied in wavelet methods for data compression, denoising, and numerical analysis of differential equations (see Daubechies 1992 [12]; Mallat 1999 [27]). Just as the sultan's single weighing uncovers the truth with optimal efficiency, a properly designed wavelet transform extracts the most relevant signal characteristics while discarding redundancies. A single, well-designed measurement is often sufficient to reveal underlying structures that are otherwise hidden. This principle holds true across contexts, whether in the allegorical setting of a sultan's court or in the modern analysis of digital signals. Mathematics, through its capacity for abstraction and generalization, consistently serves as a tool for uncovering fundamental truths across different eras and cultures.

The theory of wavelet-based operators plays a fundamental role in signal analysis and has significant applications in digital image processing. In classical Fourier analysis, signals are examined in the frequency domain by transforming them from the time domain. However, this transformation often results in the loss of time-related information, making the analysis of non-stationary signals particularly challenging. To address this limitation, various methods have been developed one of the most prominent being wavelet analysis, originally introduced by Alfred Haar. Wavelets provide a powerful tool for decomposing signals into components that are localized in both time and frequency, allowing for more precise analysis of complex signals. This talk will focus especially on recent applications of wavelet methods to two-dimensional functions and image reconstruction. We will also explore how operators defined using specific types of wavelets serve as natural extensions of classical operators and their Kantorovich-type modifications. Notably, the wavelet-based operators and the associated algorithms presented here can be seamlessly applied to real-world problems traditionally handled by Kantorovich operators, thereby enhancing both the theoretical framework and practical effectiveness of signal and image processing techniques.

Wavelet expansion, or reconstruction of signals via wavelets, allows for more accurate local identification and separation of signal features. A wavelet expansion coefficient represents a component that is itself local and easier to interpret. Wavelets can allow overlapping components of a signal to be separated in both time and frequency. Some detailed informations and advantages of the wavelets can be found in [8], [11] and [12].

In addition, we will see that the results obtained for operators defined using some special cases of wavelets represent a natural extension to the classical Bernstein operators and their Kantorovich-type modifications. It is also worth noting that the operators discussed here are closely related to hybrid type operators and quasi interpolation operators (see [3], [4] and [24]). Please also see the very recent studies of the author's on wavelet type Bezier operators, due to the advantage of the wavelet functions, which give some extensions of the previous results in the literature ([15] and [16]).

Based on the idea developed in [2], [13] and [14]-[22], the central issue of this paper is to extend the theory of interpolation to functionals and operators by introducing the two dimensional Bernstein operators  $WB_{n-1, m-1}$  by using the compactly supported Daubechies wavelets. Afterwards, we investigate the convergence problem for these operators.

It is well-known that, for a function defined on the interval  $[0, 1]$ , the Bernstein operators  $(B_n f)$ ,  $n \geq 1$ , are defined by

$$(1.1) \quad (B_n f)(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) p_{n,k}(x), \quad n \geq 1,$$

where  $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$  is the well-known Binomial distribution and called Bernstein basis ( $0 \leq x \leq 1$ ) [5]. As usual, let  $B[a, b]$  and  $C[a, b]$  be function spaces of real valued bounded and continuous functions defined on  $[a, b]$  endowed their usual norms, respectively.

We will denote by  $L^\infty(\mathbb{R})$  the space of all the essentially bounded functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  endowed with the usual ess-sup-norm. Let also  $L^p[a, b]$  ( $1 \leq p < \infty$ ) be the space of Lebesgue measurable functions defined on  $[a, b]$  with the usual  $p$ -norm.

To obtain some positive approximation results for functions in  $L^p[0, 1]$  ( $1 \leq p < \infty$ ), Kantorovich and Durrmeyer type versions of the classical Bernstein operators (1.1) were considered. For detailed approaches to this operator see the fundamental book of G.G. Lorentz [26].

Very recently, as an extension and generalization of the classical Bernstein operators, Karsli introduced in [14] the following wavelet based Bernstein operators  $WB_n : B[0, 1] \rightarrow C[0, 1]$ ,  $f \rightarrow WB_n f$ , which are given by

$$(1.2) \quad \begin{aligned} (WB_n f)(t) &:= n \sum_{k=0}^n p_{n,k}(t) \int_0^1 f(x) w(nx - k) dx \\ &= \sum_{k=0}^n p_{n,k}(t) \int_0^\lambda f\left(\frac{x+k}{n}\right) w(x) dx, \end{aligned}$$

with  $t \in [0, 1]$ , specifying that  $\text{supp}(w) \subseteq [0, \lambda]$ ,  $0 < \lambda \leq 1$ .

The author proved in [14] that the sequence  $(WB_n f)$  converges pointwise and uniformly to  $f$  on  $[0, 1]$ , and estimated the rate of these convergence results using the modulus of continuity, second order modulus of smoothness and Peetre's K-functionals. The author also obtained some convergence results in  $L^p$  spaces via these operators.

In his Ph.D. thesis [6] written under the direction of G.G. Lorentz and afterwards in the paper [7], the famous German mathematician P.L. Butzer considered two dimensional Bernstein polynomials on the square  $\square := \{(x, y) : 0 \leq x, y \leq 1\}$  as follows:

$$B_{n,m}(f; x, y) = \sum_{k=0}^n \sum_{j=0}^m f\left(\frac{k}{n}, \frac{j}{m}\right) p_{n,k}(x) p_{m,j}(y),$$

where  $p_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}$ .

In the present study, we will reconstruct bivariate Bernstein operators, where location and time are very important and effective, with the help of wavelet expansions. Moreover, we will examine and analyse various properties of the wavelet based extension of the operators. Afterwards, we will provide some examples both on the convergence of functions of two variables and on image processing applications.

## 2. PRELIMINARIES AND AUXILIARY RESULTS

Let us consider two orthogonal functions: the scaling function (or father wavelet)  $\phi(t)$  and the wavelet function (or mother wavelet)  $\psi(t)$ . By scaling and translation of these two orthogonal functions we obtain a complete basis set. These functions have the following important properties;

$$\int_{-\infty}^{\infty} \phi(t)dt = 1, \quad \int_{-\infty}^{\infty} \psi(t)dt = 0,$$

$\phi, \psi \in L^2(\mathbb{R})$ , and orthogonal (see [11, 12]). In general, the wavelets refer to the set of family of orthonormal functions of the form

$$(2.3) \quad \psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right), \quad a > 0, b \in \mathbb{R},$$

where  $\psi$  is the basic (mother) wavelet. The simplest wavelet is known as the Haar wavelet given by:

$$\psi(x) = \begin{cases} 1 & , \quad 0 \leq x < \frac{1}{2} \\ -1 & , \quad \frac{1}{2} \leq x < 1 \\ 0 & , \quad e.w. \end{cases}$$

with the corresponding scaling function (father wavelet)

$$\phi(t) = \begin{cases} 1 & , \quad 0 \leq x < 1 \\ 0 & , \quad e.w. \end{cases}.$$

Haar wavelets constitutes an orthonormal system for the space of square-integrable functions on the real line.

We now consider a special orthonormal bases, called wavelets. There is a scaling function (father wavelet)  $\phi(t)$  with  $\{\phi(t-n)\}$  are orthogonal and the mother wavelet  $\psi(t)$  based on the father wavelet  $\phi(t)$  gives rise to the orthonormal basis

$$(2.4) \quad \psi_{j,k}(t) := 2^{j/2}\psi(2^j t - k)$$

of  $L^2(\mathbb{R})$ . Moreover, a multiresolution analysis (MRA) is a sequence  $(V_j)_{j \in \mathbb{Z}}$  of closed subspaces of  $L^2(\mathbb{R})$ , whose elements are scaling functions (father wavelets).

It is well-known that, each  $f \in L^2(\mathbb{R})$  has the following representation

$$f(x) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} b_{j,k} \psi_{j,k}(x),$$

called wavelet expansion and  $b_{j,k}$  are wavelet coefficients given by

$$b_{j,k} = \langle f(x), \psi_{j,k}(x) \rangle = 2^{j/2} \int_{\mathbb{R}} f(x) \overline{\psi(2^j x - k)} dx$$

(see [2, 13, 23, 25] and [28]).

Let us assume that father wavelets  $w \in L_{\infty}(\mathbb{R})$  satisfies:

- w<sub>1</sub>)** There is a real constant  $0 < \lambda \leq 1$  such that  $\text{supp } w \subset [0, \lambda]$ ,
- w<sub>2</sub>)**

$$\int_{\mathbb{R}} w(x) dx = 1,$$

w<sub>3</sub>) The first  $N$  moments satisfy

$$m_j^w(w) := \int_{\mathbb{R}} x^j w(x) dx = 0, \quad j = 1, \dots, N.$$

Obviously, the absolute moments of the father wavelet  $w$

$$M_j^w(w) := \int_{\mathbb{R}} |x|^j |w(x)| dx < +\infty$$

for every  $j \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$ .

Wavelets that meet the above conditions are called compactly supported Daubechies wavelets. Daubechies wavelets have strong relations with the properties of continuity and differentiability. They are supported with  $[0, 2N - 1]$ , in addition there exists a constant  $r > 0$  such that for  $N \geq 2$ ,  $w \in C^{rN}(\mathbb{R})$  and have a given number of vanishing moments. When  $N = 1$ , then the first Daubechies wavelet  $\psi$  will be the classical Haar basis. As  $N$  increases, the regularity of the wavelet increases (see [11, 12]). This means that if we want to use Daubechies wavelets to reconstruct a function, it is more convenient to choose or construct wavelets based on the continuity or differentiability properties of the given function.

Owing to the above definitions, first of all we introduce the bivariate case of the wavelet type Bernstein operators  $WB_n$ . We denote

$$(2.5) \quad \tilde{f}(z, y) := \begin{cases} f(z, y), & (z, y) \in [0, 1]^2 \\ 0, & e.w. \end{cases}$$

Let  $B([0, 1]^2) = \{f : [0, 1]^2 \rightarrow \mathbb{R} \mid f \text{ be bounded on } [0, 1]^2\}$ . The norm on  $B([0, 1]^2)$  is given by

$$\|f\|_2 := \sup_{(x,t) \in [0,1]^2} |f(x,t)|.$$

**Definition 2.1.** Let  $f \in B([0, 1]^2)$ , and let  $w \in L_\infty(\mathbb{R})$  be a father wavelet satisfying (w<sub>1</sub>), (w<sub>2</sub>) and (w<sub>3</sub>). Then the bivariate wavelet type Bernstein operators are defined by:

$$(2.6) \quad \begin{aligned} (WB_{n-1, m-1}f)(x, t) &= nm \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1, k}(t) p_{m-1, j}(x) \int_0^1 \int_0^1 f(z, y) w(mz - j) w(ny - k) dz dy \\ &= nm \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1, k}(t) p_{m-1, j}(x) \int_{\mathbb{R}} \int_{\mathbb{R}} \tilde{f}(z, y) w(mz - j) w(ny - k) dz dy, \end{aligned}$$

where the function  $\tilde{f}$  being defined in (2.5), with  $t, x \in [0, 1]$ , specifying that  $\text{supp}(w) \subseteq [0, \lambda]$ ,  $0 < \lambda \leq 1$ .

**Remark 2.1.** If we choose the father wavelet  $w$  as the Haar scaling function, namely  $w(\cdot) = \chi_{[0,1]}(\cdot)$ , then clearly our wavelet type operators reduce to the Kantorovich form of the Bernstein operators. Indeed:

$$\begin{aligned} (WB_{n-1, m-1}f)(x, t) &= nm \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1, k}(t) p_{m-1, j}(x) \int_0^1 \int_0^1 f(z, y) w(mz - j) w(ny - k) dz dy \\ &= nm \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1, k}(t) p_{m-1, j}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{\frac{j}{m}}^{\frac{j+1}{m}} f(z, y) dz dy = (K_{n, m}f)(x, t). \end{aligned}$$

This shows that the bivariate wavelet Bernstein operators (2.6) are a natural extension of the bivariate Bernstein-Kantorovich operators.

### 3. FUNDAMENTAL PROPERTIES

The following approximate results for the wavelet type Bernstein operators need to be remembered for dealing with application and reconstruction of functions. In particular, the following convergence theorem applies when continuous signals (functions) are considered.

Throughout, we denote the monomials by

$$(3.7) \quad e_{s,l} := e_{s,l}(z, y) = \begin{cases} z^s y^l, & (z, y) \in [0, 1]^2 \\ 0 & , \text{ e.w.} \end{cases}$$

for  $s, l = 0, 1, 2, \dots$ . We have the followings.

**Theorem 3.1.** *Let  $w \in L_\infty(\mathbb{R})$  be a father wavelet satisfies  $w_1, w_2$  and  $w_3$ . Then the moments of bivariate wavelet Bernstein operators, constructed by using the compactly supported Daubechies wavelets (2.6) and the Bernstein operators (1.1) are the same, namely*

$$\begin{aligned} (WB_{n-1, m-1} e_{s,0})(x, t) &= (B_{m-1} e_{s,0})(x) \\ (WB_{n-1, m-1} e_{0,l})(x, t) &= (B_{n-1} e_{0,l})(t) \\ (WB_{n-1, m-1} (e_{0,l} + e_{s,0}))(x, t) &= (B_{n-1} e_{0,l})(t) + (B_{m-1} e_{s,0})(x), \end{aligned}$$

hold true.

*Proof.* In view of the definition of the operator (2.6), we have

$$\begin{aligned} &(WB_{n-1, m-1} e_{s,0})(x, t) \\ &= nm \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1, k}(t) p_{m-1, j}(x) \int_0^1 \int_0^1 z^s w(mz - j) w(ny - k) dz dy \\ &= n \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1, k}(t) p_{m-1, j}(x) \int_{\mathbb{R}} \int_{\mathbb{R}} \left( \frac{u+j}{m} \right)^s w(u) w(ny - k) du dy \\ &= \frac{n}{m^s} \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1, k}(t) p_{m-1, j}(x) \int_{\mathbb{R}} \int_{\mathbb{R}} (u+j)^s w(u) w(ny - k) du dy \\ &= \frac{n}{m^s} \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1, k}(t) p_{m-1, j}(x) \int_{\mathbb{R}} \int_{\mathbb{R}} \left[ \sum_{i=0}^s \binom{s}{i} u^i j^{s-i} \right] w(u) w(ny - k) du dy. \end{aligned}$$

In view of  $w_3$ , one has for  $i \neq 0$

$$\int_{\mathbb{R}} \left[ \sum_{i=0}^s \binom{s}{i} u^i j^{s-i} \right] w(u) du = 0$$

and for  $i = 0$  and from  $\mathbf{w}_2$  we get

$$\begin{aligned}
 (WB_{n-1,m-1}e_{s,0})(x,t) &= \frac{n}{m^s} \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t)p_{m-1,j}(x) \int_{\mathbb{R}} \int_{\mathbb{R}} j^s w(u)w(ny-k)du dy \\
 &= \frac{n}{m^s} \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t)p_{m-1,j}(x) \int_{\mathbb{R}} \int_{\mathbb{R}} j^s w(u)w(ny-k)du dy \\
 &= \sum_{j=0}^{m-1} \frac{j^s}{m^s} p_{m-1,j}(x) \\
 &= (B_{m-1}e_{s,0})(x) = x^s.
 \end{aligned}$$

The proof of the remain parts are similar, so we omit.  $\square$

**Remark 3.2.** Moreover, the central moments of the bivariate wavelet type Bernstein operators (2.6) are the same as of the classical Bernstein operators (1.2). Indeed, as in the previous Theorem 3.1, we get

$$\begin{aligned}
 &(WB_{n-1,m-1}(e_{1,0}-x)^\beta)(x,t) \\
 &= nm \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t)p_{m-1,j}(x) \int_0^1 \int_0^1 (z-x)^\beta w(mz-j)w(ny-k)dz dy \\
 &= m \sum_{j=0}^{m-1} p_{m-1,j}(x) \int_{\mathbb{R}} (z-x)^\beta w(mz-j)dz \\
 &= \sum_{j=0}^{m-1} p_{m-1,j}(x) \int_{\mathbb{R}} \left(\frac{u+j}{m}-x\right)^\beta w(u)du \\
 &= \frac{1}{m^\beta} \sum_{j=0}^{m-1} p_{m-1,j}(x) \int_{\mathbb{R}} (u+j-mx)^\beta w(u)du \\
 &= \frac{1}{m^\beta} \sum_{j=0}^{m-1} p_{m-1,j}(x) \int_{\mathbb{R}} \left[ \sum_{i=0}^{\beta} \binom{\beta}{i} u^i (j-mx)^{\beta-i} \right] w(u)du.
 \end{aligned}$$

Again by the properties of the compactly supported Daubechies wavelets, namely  $\mathbf{w}_2$  and  $\mathbf{w}_3$ , we get

$$\begin{aligned}
 (WB_{n-1,m-1}(e_{1,0}-x)^\beta)(x,t) &= \frac{1}{m^\beta} \sum_{j=0}^{m-1} p_{m-1,j}(x) (j-mx)^\beta \\
 &= (B_{m-1}(e_{1,0}-x)^\beta)(x).
 \end{aligned}$$

Similarly one has

$$\begin{aligned}
 (WB_{n-1,m-1}(e_{0,1}-t)^l)(x,t) &= \frac{1}{n^l} \sum_{k=0}^{n-1} p_{n-1,k}(x) (k-nt)^l \\
 &= (B_{n-1}(e_{0,1}-t)^l)(t).
 \end{aligned}$$

Throughout this work, as in the case of the Bernstein operators, we assume that, the first two central moments of the Bernstein operators, constructed by using the compactly supported

Daubechies wavelets (2.6) satisfy

$$(3.8) \quad \begin{aligned} m_0^{l=0} &:= (WB_{n-1,m-1}(e_{0,1}-t)^0)(x,t) = 1, \\ m_0^{s=0} &:= (WB_{n-1,m-1}(e_{1,0}-x)^0)(x,t) = 1, \\ m_1^{l=1} &:= (WB_{n-1,m-1}(e_{0,1}-t)^1)(x,t) = 0, \\ m_1^{s=1} &:= (WB_{n-1,m-1}(e_{1,0}-x)^1)(x,t) = 0, \\ m_1^{l=2} &:= (WB_{n-1,m-1}(e_{0,1}-t)^2)(x,t) = \frac{t(1-t)}{n} \\ m_1^{s=2} &:= (WB_{n-1,m-1}(e_{1,0}-x)^2)(x,t) = \frac{x(1-x)}{m} \end{aligned}$$

and

$$(3.9) \quad \begin{aligned} m_1^{l=2} &= \frac{t(1-t)}{n} \leq \frac{1}{4n} \\ m_1^{s=2} &= \frac{x(1-x)}{m} \leq \frac{1}{4m} \end{aligned}$$

for every  $t, x \in [0, 1]$ .

It is also well-known that for each  $i \in \mathbb{N}_0$  there is a constant  $A_i$  only depending upon  $s$  or  $l$  such that

$$\begin{aligned} 0 \leq m_1^{l=2i} &\leq \frac{A_i}{n^i} < \infty, \\ 0 \leq m_1^{s=2i} &\leq \frac{A_i}{m^i} < \infty \end{aligned}$$

hold (page 15 eq (6) Lorentz [26], see also [1]).

Moreover, for every  $t \in [0, 1]$  and for some  $\beta > 0$ , the discrete absolute moments of order  $\beta$  satisfy

$$(3.10) \quad (B_n |x-t|^\beta)(t) \leq 2\Gamma\left(\frac{\beta}{2} + 1\right) \frac{1}{n^{\beta/2}} < \infty,$$

where  $\Gamma(\bullet)$  stands for the Gamma function (see [1]).

#### 4. CONVERGENCE PROPERTIES

We now introduce some notations and structural hypotheses, which will be fundamental in proving our convergence theorems. This section also provides the main approximation results of the paper.

To address this task, it is necessary to recall the notions of the modulus of continuities of a given bivariate functions.

**Definition 4.2.** Let  $f \in C([a, b]^2)$  and  $\delta > 0$  be given. Then the complete modulus of continuity is given by;

$$\omega(\delta) = \sup_{\sqrt{(u_1-u_2)^2+(v_1-v_2)^2} \leq \delta} |f(u_1, v_1) - f(u_2, v_2)|.$$

Further on, the first and second partial modulus of continuity are given by

$$\begin{aligned} \omega_1(\delta_1, 0) &= \sup_{|u_1-u_2| \leq \delta_1} |f(u_1, v_1) - f(u_2, v_1)|, \\ \omega_2(0, \eta) &= \sup_{|v_1-v_2| \leq \eta} |f(u_1, v_1) - f(u_1, v_2)|. \end{aligned}$$

**Definition 2.** Recall that the function  $\omega(f; \delta)$  has the following well-known properties;

- (i) Let  $\lambda \in \mathbb{R}^+$ , then  $\omega(f; \lambda\delta) \leq (\lambda + 1)\omega(f; \delta)$ ,
- (ii)  $\lim_{\delta \rightarrow 0^+} \omega(f; \delta) = 0$ ,
- (iii)  $|f(u_1, v_1) - f(u_2, v_2)| \leq \omega(\delta) \left(1 + \frac{\sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2}}{\delta}\right)$ .

Note that the same properties also hold for partial moduli of continuity.

We are now ready to establish one of the first main results of this study, which gives a strong relation between Bernstein operators (1.2) and our new operators (2.6) constructed via wavelets. We have the following result.

**Theorem 4.2.** Let  $f \in B([0, 1]^2)$  be a measurable function and let  $\psi \in L_\infty(\mathbb{R})$  be a father wavelet satisfies  $\mathbf{w}_1$ ,  $\mathbf{w}_2$  and  $\mathbf{w}_3$ . Then

$$\lim_{(n,m) \rightarrow \infty} (WB_{n-1,m-1}f)(x_0, t_0) = f(x_0, t_0)$$

holds true at each point  $(x_0, t_0)$  of continuity of  $f$ .

*Proof.* In view of the definition of the operator (2.6), one has

$$\begin{aligned} & (WB_{n-1,m-1}f)(x_0, t_0) - f(x_0, t_0) \\ &= nm \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t_0) p_{m-1,j}(x_0) \int_0^1 \int_0^1 f(z, y) w(mz - j) w(ny - k) dz dy - f(x_0, t_0). \end{aligned}$$

By Theorem 3.1, we know that

$$(WB_{n-1,m-1}1)(x, t) = 1,$$

and hence, we can write

$$\begin{aligned} & |(WB_{n-1,m-1}f)(x_0, t_0) - f(x_0, t_0)| \\ &= \left| nm \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t_0) p_{m-1,j}(x_0) \int_0^1 \int_0^1 [f(z, y) - f(x_0, t_0)] w(mz - j) w(ny - k) dz dy \right| \\ &\leq \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t_0) p_{m-1,j}(x_0) \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \tilde{f}\left(\frac{u+j}{m}, \frac{v+k}{n}\right) - f(x_0, t_0) \right| w(u) w(v) du dv. \end{aligned}$$

Let us divide the last term into two parts as;

$$|(WB_{n-1,m-1}f)(x_0, t_0) - f(x_0, t_0)| \leq P_1 + P_2,$$

where

$$\begin{aligned} P_1 &= \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t_0) p_{m-1,j}(x_0) \\ &\times \iint_{\sqrt{\left(\frac{u+j}{m} - x_0\right)^2 + \left(\frac{v+k}{n} - t_0\right)^2} < \delta} \left| \tilde{f}\left(\frac{u+j}{m}, \frac{v+k}{n}\right) - f(x_0, t_0) \right| w(u) w(v) du dv \end{aligned}$$

and

$$P_2 = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t_0) p_{m-1,j}(x_0) \\ \times \iint_{\sqrt{\left(\frac{u+j}{m}-x_0\right)^2 + \left(\frac{v+k}{n}-t_0\right)^2} \geq \delta} \left| \tilde{f}\left(\frac{u+j}{m}, \frac{v+k}{n}\right) - f(x_0, t_0) \right| w(u)w(v) dudv$$

Since  $(x_0, t_0)$  is a continuity point of  $f$ , then clearly

$$|f(z, y) - f(x_0, t_0)| < \epsilon$$

whenever  $\sqrt{(z-x_0)^2 + (y-t_0)^2} < \delta$ , hence we can write

$$P_1 < \epsilon.$$

On the other hand, since

$$|f(z, y) - f(x_0, t_0)| \leq 2 \|f\|_2$$

whenever  $\sqrt{(z-x_0)^2 + (y-t_0)^2} \geq \delta$ , we get

$$P_2 = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t_0) p_{m-1,j}(x_0) \\ \times \iint_{\sqrt{\left(\frac{u+j}{m}-x_0\right)^2 + \left(\frac{v+k}{n}-t_0\right)^2} \geq \delta} \left| \tilde{f}\left(\frac{u+j}{m}, \frac{v+k}{n}\right) - f(x_0, t_0) \right| w(u)w(v) dudv \\ \leq 2 \|f\|_2 \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t_0) p_{m-1,j}(x_0) \iint_{\sqrt{\left(\frac{u+j}{m}-x_0\right)^2 + \left(\frac{v+k}{n}-t_0\right)^2} \geq \delta} w(u)w(v) dudv \\ \leq 2 \|f\|_2 \frac{m_1^{l=2} + m_1^{s=2}}{\delta^2} \leq \frac{\|f\|}{\delta^2} \left( \frac{1}{2n} + \frac{1}{2m} \right).$$

Collecting these estimates, we have

$$\lim_{(n,m) \rightarrow \infty} (WB_{n-1,m-1}f)(x_0, t_0) = f(x_0, t_0).$$

This completes the proof.  $\square$

As a consequence of the Theorem 3.1 we have also the following uniform convergence result.

**Corollary 4.1.** *The same arguments apply to the case when  $f \in C([0, 1]^2)$ . In this case the convergence is uniform with respect to  $x, t \in [0, 1]$ , and hence one has*

$$\lim_{(n,m) \rightarrow \infty} \|(WB_{n-1,m-1}f) - f\|_{C([0,1]^2)} = 0.$$

## 5. PRACTICAL EXAMPLES, GRAPHICAL REPRESENTATIONS

**5.1. Application to functions.** Now, we will give some graphical examples for these approach, namely convergence to functions by means of wavelet based bivariate Bernstein operators  $(WB_{n-1,m-1}f)(x, y)$ .

**Example 5.1.** Let  $f(x, y) = 3(\sin(10xy) + 1)$ . We consider a special case of the wavelet based Bernstein operators  $(WB_{n-1,m-1}f)(x, y)$  constructed by using Shannon wavelet function. Then one has for  $n, m = 2, 5, 10, 20$  and for 40.

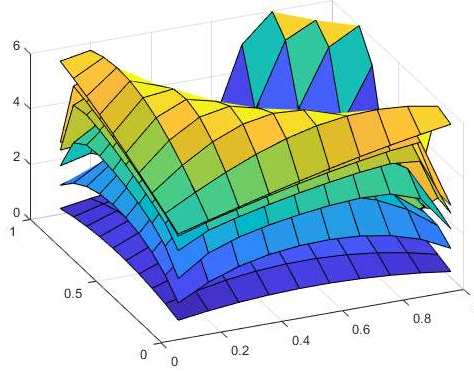


FIGURE 1. Approximation to  $f(x, y) = 3(\sin(10xy) + 1)$  by Shannon wavelet based Bernstein operator, for  $n, m = 2, 5, 10, 20$  and 40.

**Example 5.2.** Let  $f(x, y) = 3(\sin(10xy) + 1)$ . We consider the wavelet based Bernstein operators  $(WB_{n-1,m-1}f)(x, y)$  constructed by using Haar scaling function. Then one has for  $n, m = 20, 25$  and for 30.

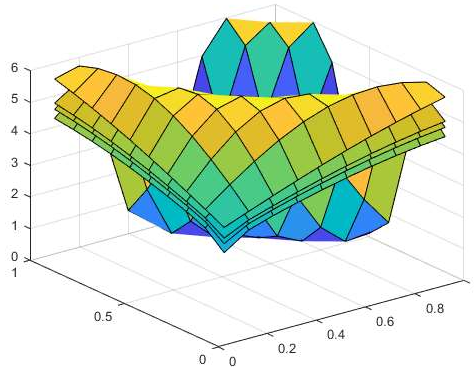


FIGURE 2. Approximation of  $f(x, y) = 3(\sin(10xy) + 1)$  by Haar wavelet based Bernstein operator, for  $n, m = 20, 25$  and 30.

**Example 5.3.** Let  $f(x, y) = 3(\sin(10xy) + 1)$ , and we consider the wavelet based Sampling operators  $(WS_{n,m}f)(x, y)$  constructed by using Shannon wavelet function. Then one has for  $n, m = 10, 30$  and for 50.

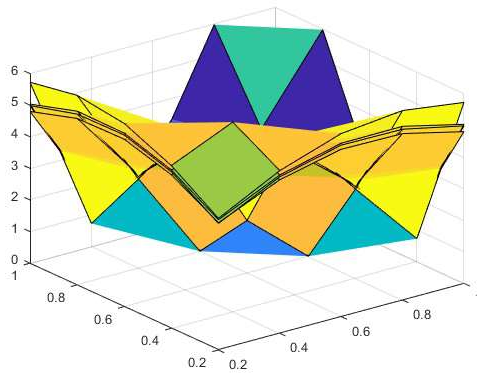


FIGURE 3. Approximation to  $f(x, y) = 3(\sin(10xy) + 1)$  by Shannon wavelet based Sampling operator, for  $n, m = 10, 30$  and  $50$ .

**Example 5.4.** Let  $f(x, y) = x^2 - x + 1$ . We consider a special case of the wavelet based Bernstein operators  $(WB_{n-1, m-1}f)(x, y)$  constructed by using Shannon wavelet function. Then one has for  $n, m = 2, 5, 10, 20$  and for  $40$ .

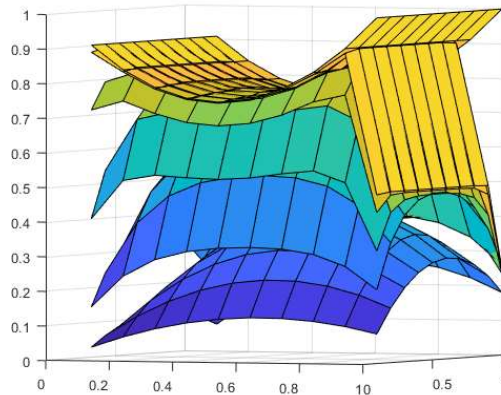


FIGURE 4. Approximation to  $f(x, y) = x^2 - x + 1$  by Shannon wavelet based Bernstein operator, for  $n, m = 2, 5, 10, 20$  and for  $40$ .

**Example 5.5.** Let  $f(x, y) = x^2 - x + 1$ , and we consider the wavelet based Bernstein operators  $(WB_{n-1, m-1}f)(x, y)$  constructed by using Haar scaling function. Then one has for  $n, m = 2, 5, 10, 20$  and for  $40$ .

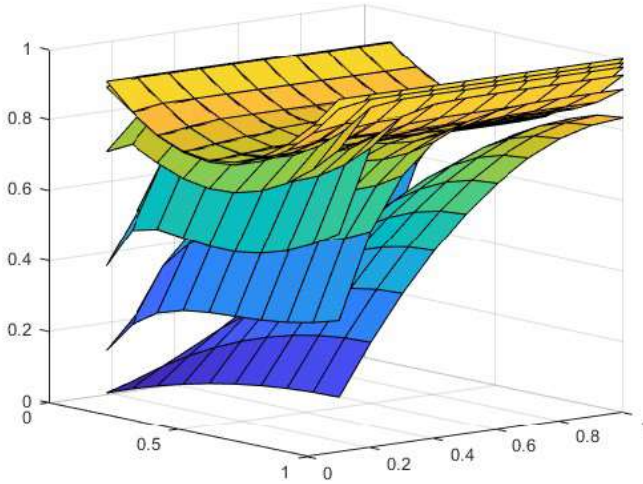


FIGURE 5. Approximation to  $f(x, y) = x^2 - x + 1$  by Haar wavelet based Bernstein operator, for  $n = 2, 5, 10, 20$  and  $40$ .

**5.2. Application on images.** As modern technology has advanced, enlarging digital images has become widespread in various fields including digital photography, medical imaging, and smartphones.

Numerous zooming techniques have been explored in the literature, such as pixel duplication, interpolation methods, zero-order hold, and others.

Now, we will give some image reconstruction examples via wavelet based operators.

Starting from the  $64 \times 64$  pixel grayscale Baboon image:

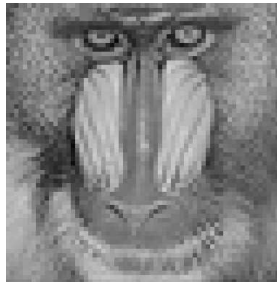


FIGURE 6. Original Baboon Image ( $64 \times 64$ ) pixels

The image was first downsampled to  $32 \times 32$  pixels.

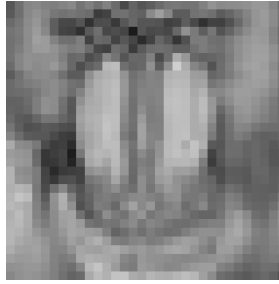


FIGURE 7. Downscaled Baboon Image ( $32 \times 32$ ) pixels

In order to demonstrate the smoothing capabilities of the aforementioned operators, we will reconstruct the  $32 \times 32$  pixel downscaled Baboon image back to its original  $64 \times 64$  resolution using Haar wavelet-based Bernstein and Shannon wavelet-based Bernstein operators.

The quality of the reconstructed images was assessed by calculating the Peak Signal-to-Noise Ratio (PSNR) with respect to the original image.

The aforementioned procedures were implemented using MATLAB programming language.

We consider the wavelet based Bernstein operators  $(WB_{n-1,m-1}f)(x,y)$  constructed by using Haar scaling function. Then one has for  $n, m = 1$ ,

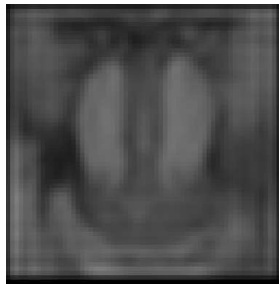


FIGURE 8. Reconstruction Baboon Image ( $64 \times 64$  pixels) via Haar based Bernstein operators for  $n, m = 1$ .

In this reconstruction the Peak Signal-to-Noise Ratio (PSNR) with respect to the original image is **16.0959**.

We consider the wavelet based Bernstein operators  $(WB_{n-1,m-1}f)(x,y)$  constructed by using Haar scaling function. Then one has for  $n, m = 5$ ,

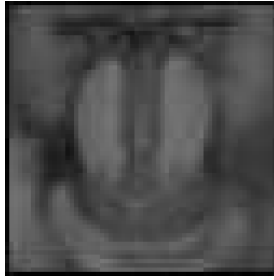


FIGURE 9. Reconstruction Baboon Image ( $64 \times 64$  pixels) via Haar based Bernstein operators for  $n, m = 5$ .

In this reconstruction the Peak Signal-to-Noise Ratio (**PSNR**) with respect to the original image is **19.3205**.

We consider the wavelet based Bernstein operators  $(WB_{n-1, m-1}f)(x, y)$  constructed by using Shannon wavelet function. Then one has for  $n, m = 1$ ,

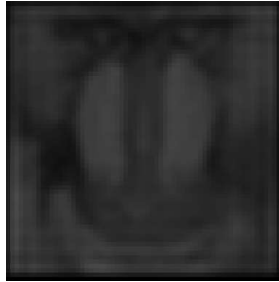


FIGURE 10. Reconstruction Baboon Image ( $64 \times 64$  pixels) via Shannon based Bernstein operators for  $n, m = 1$ .

In this reconstruction the Peak Signal-to-Noise Ratio (**PSNR**) with respect to the original image is **14.9116**.

Starting from the  $64 \times 64$  pixel grayscale Umut's image:



FIGURE 11. Original Umut's Image ( $64 \times 64$ ) pixels

The image was first downsampled to  $32 \times 32$  pixels.



FIGURE 12. Downscaled Umut's Image ( $32 \times 32$ ) pixels

We consider the wavelet based Bernstein operators  $(WB_{n-1,m-1}f)(x,y)$  constructed by using Haar scaling function. Then one has for  $n, m = 1$ ,



FIGURE 13. Reconstruction Umut's Image ( $64 \times 64$  pixels) via Haar based Bernstein operators for  $n, m = 1$ .

In this reconstruction the Peak Signal-to-Noise Ratio (PSNR) with respect to the original image is **16.1996**.

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