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Editorial Note

International Workshop on Modern Problems of Analysis, Optimization, Approximation and Their Applications (IWMPAOATA)

ROMAN DMYTRYSHYN*

ABSTRACT. The guest editor provides an overview of the International Workshop on Modern Problems of Analysis, Optimization, Approximation and Their Applications (IWMPAOATA), held on 25–27 June 2025 at International Telematic University UNINETTUNO, Rome, Italy, as part of the Erasmus+ program, summarizing its objectives, scope, scientific aims, and the key highlights of the event. The workshop brought together Italian, Ukrainian and international mathematicians who presented results on such scientific topics as linear and nonlinear functional analysis, function theory, approximation theory, numerical analysis, and optimization theory. The note briefly introduces the papers included in this special issue of the ALTAY Conference Proceedings in Mathematics, which reflect current advances in mathematics.

1. REPORT ON THE CONFERENCE

The International Workshop on Modern Problems of Analysis, Optimization, Approximation and Their Applications (IWMPAOATA) was held on 25–27 June 2025 at International Telematic University UNINETTUNO, Rome, Italy. The event provided insight into scientific collaboration between Italy and Ukraine, and also highlighted Europe's support for cooperation through the Erasmus+ program. In total, 75 participants from 8 countries took part in the workshop, including 49 speakers.

The scientific program covered a wide range of topics, including linear and nonlinear functional analysis, function theory, approximation theory, numerical analysis, optimization theory, and their applications to real-world problems.

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The event was hosted by Prof. Clemente Cesarano and Prof. Roman Dmytryshyn with the support of the Organizing and Scientific Committees. Further details, including the complete program and abstracts, are available on the official website: `https://sites.google.com/uninettunouniversity.net/iwmpaoata/home?authuser=0`

2. Introducing the Special Issue

The papers published in this special issue of the ALTAY Conference Proceedings in Mathematics originate from the International Workshop on Modern Problems of Analysis, Optimization, Approximation and Their Applications (IWMPAOATA), held on 25–27 June 2025 at International Telematic University UNINETTUNO, Rome, Italy.

Following the workshop, the participants were invited to submit extended versions of their presented works to this special issue. After a rigorous peer-review process, the following papers have been accepted for publication:

- *Computational aspects of approximating the Horn hypergeometric functions H_3 by branched continued fractions*, by Marta Dmytryshyn, Sofiia Hladun, Mykhailo Holod, and Volodymyr Hladun. This paper investigates the approximation of the Horn hypergeometric function H_3 using its expansion into the branched continued fraction.
- *Mathematical modelling of control-loss detection via risk-sensitive reinforcement learning on partially observable Markov decision processes*, by Oleksandr Chaban and Volodymyr Hladun. The paper presents a method for identifying high-risk loss-of-control episodes in digital settings by combining risk-sensitive reinforcement learning with decision-making under partial observability.
- *Truncation error bounds of branched continued fraction expansions of special ratios of Horn's hypergeometric functions H_4*, by Roman Dmytryshyn, Clemente Cesarano, and Ilona-Anna Lutsiv. The paper establishes the truncation error bounds for the branched continued fraction extensions of some Horn's hypergeometric function H_4 ratios with real parameters and variables.
- *Maximum modulus of slice entire regular functions of quaternionic variable with bounded index*, by Vita Baksa, Andriy Bandura, and Oleh Skaskiv. The paper contains new results describing local behavior of slice entire regular functions of quaternionic variable.
- *About Borel type relation for some positive integrals*, by Andriy Bandura, Andriy Bodnarchuk, and Oleh Skaskiv. The paper describes asymptotic behavior of functions which are represented by integrals of the form
$$F(x) = \int_0^{+\infty} a(t)f(x+t)\nu(dt),$$
where ν is locally finite measure on $\mathbb{R}_+$, a is positive ν-measurable function, f is positive and increasing to $+\infty$ in $[0,+\infty)$ function such that $f(0)=1$ and $\ln f(x)$ is a convex on the interval $[0,+\infty)$ function.
- *Approximation characteristics of Stepanets–Orlicz type spaces*, by Andrii Shydlich. The paper presents the spaces $\mathcal{S}_{M,\Phi}$ and shows their connection with the well-known Stepanets spaces $\mathcal{S}^p_\Phi$, Orlicz spaces $\mathcal{S}_M$, and others.

3. Acknowledgements

We express our sincere gratitude to all members of the Organizing and Scientific Committees for their efforts in making the conference a success. Special thanks are extended to all reviewers for their valuable time and constructive comments, which significantly improved the quality of the published papers.

The Editor also wish to thank the ALTAY editorial office for their professional support during the preparation of this issue.

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Research Article

Computational aspects of approximating the Horn hypergeometric functions H_3 by branched continued fractions

MARTA DMYTRYSHYN*, SOFIIA HLADUN, MYKHAILO HOLOD, AND VOLODYMYR HLADUN

ABSTRACT. This paper investigates the approximation of Horn hypergeometric function H_3 using branched continued fractions. Based on the formal branched continued fraction expansion for the ratio of hypergeometric functions H_3, a branched continued fraction expansion for a specific function is constructed. Numerical experiments using a custom Python implementation compare the convergence properties of the branched continued fraction approximants with the partial sums of the corresponding double power series. Results, presented in tables and plots, demonstrate that the branched continued fraction approach generally offers better convergence properties, including potentially wider regions of convergence and higher accuracy, particularly in regions where the power series diverges or converges slowly. The convergence behavior is visualized through error plots in different complex planes, suggesting that the branched continued fraction provides a robust tool for approximating this special function. Additionally, algorithms for computing approximants of continued fractions are studied. The results show that the continuant method is unstable and slower than the backward recurrence algorithms. The backward recurrence algorithms are stable, and their parallel implementation is faster than the single-threaded version.

Keywords: Horn hypergeometric function H_3, branched continued fraction, convergence, approximation by rational functions, backward recurrence algorithm.

2020 Mathematics Subject Classification: 33C65, 30B99, 40A99, 41A20, 65D15.

1. INTRODUCTION

Current practices in applying modern technology and science in security and defense demand deep knowledge of applied mathematics, particularly special functions. These functions are used to construct approximate or exact analytical solutions to equations describing complex processes, such as those in physics, chemistry, and engineering, thereby providing a better and more meaningful understanding of the properties of these processes and mechanisms. Due to their importance, many works are dedicated to these functions, and even website was developed (https://functions.wolfram.com).

Branched Continued Fractions (BCFs) are a natural generalization of classical continued fractions [9, 10, 11, 13, 26, 27], inheriting many of their properties, including the most important approximation characteristics: a wide convergence region, good convergence rate [7, 12, 14, 16, 25], and numerical stability, which involves no accumulation or slow accumulation of errors when computing approximants [15, 20, 21, 22, 23]. Dmytro Bodnar and his students have demonstrated the effectiveness of approximating special functions, particularly hypergeometric functions, using BCFs [1, 2, 4, 5] and [18, 19, 28].

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Despite significant achievements in approximating special functions, this topic remains one of the most important in the analytical theory of BCFs and still has many open problems, especially concerning hypergeometric functions.

One of the main tasks is investigating the convergence and numerical stability of BCF expansions for hypergeometric functions. Given the small number of theoretical studies regarding their convergence, applied research into these expansions, particularly for the Horn function H_3, is an interesting direction that should be explored to bring new knowledge to the study of special functions.

2. Branched continued fraction expansion

Let us consider Horn's hypergeometric function H_3 [24], defined as a double power series (DPS) of the form

$$H_3(\alpha,\beta;\gamma;\mathbf{z})=\sum_{r,s=0}^{+\infty}\frac{(\alpha)_{2r+s}(\beta)_s}{(\gamma)_{r+s}}\frac{z_1^r z_2^s}{r!s!}, \tag{2.1}$$

where $\alpha,\beta,\gamma\in\mathbb{C}$, $\gamma\notin\{0,-1,-2,\ldots\}$, $\mathbf{z}=(z_1,z_2)\in\mathbb{C}^2$, $(x)_k$ is the Pochhammer symbol, and let I be a set of multiindices

$$I=\{i(k)=(i_1,i_2,\ldots,i_k):\ 1\le i_r\le 2,\ 1\le r\le k,\ k\ge 1\}.$$

In [3], a formal BCF expansion for the ratio of functions (2.1) was obtained:

$$\frac{H_3(\alpha,\beta;\gamma;\mathbf{z})}{H_3(\alpha+1,\beta;\gamma+1;\mathbf{z})}=1+\sum_{i_1=1}^{2}\cfrac{c_{i(1)}(\mathbf{z})}{d_{i(1)}(\mathbf{z})+\sum\limits_{i_2=1}^{2}\cfrac{c_{i(2)}(\mathbf{z})}{d_{i(2)}(\mathbf{z})+\ddots}}, \tag{2.2}$$

where the elements are determined by the formulas

$$c_1(\mathbf{z})=-\frac{(2\gamma-\alpha)(\alpha+1)z_1}{\gamma(\gamma+1)},\quad c_2(\mathbf{z})=-\frac{\beta(\gamma-\alpha)(1-4z_1)z_2}{\gamma(\gamma+1)}, \tag{2.3}$$

$$d_{i(k)}(\mathbf{z})=1-\frac{\alpha-\beta-1+\sum_{r=0}^{k-1}(\delta_{i_r}^1-\delta_{i_r}^2)}{\gamma+k}\delta_{i_k}^2 z_2-2\frac{2\gamma-\alpha+k+\sum_{r=0}^{k-1}\delta_{i_r}^2}{\gamma+k}\delta_{i_k}^2 z_1, \tag{2.4}$$

$$c_{i(k),1}(\mathbf{z})=-\frac{(2\gamma-\alpha+k+\sum_{r=0}^{k}\delta_{i_r}^2-2\delta_{i_k}^2(2\gamma-\alpha-\beta+k)z_2)(\alpha+\sum_{r=0}^{k}\delta_{i_r}^1)z_1}{(\gamma+k)(\gamma+k+1)}, \tag{2.5}$$

$$c_{i(k),2}(\mathbf{z})=-\frac{(\beta+\sum_{r=0}^{k}\delta_{i_r}^2)(\gamma-\alpha+\sum_{r=0}^{k}\delta_{i_r}^2)(1-4z_1)z_2}{(\gamma+k)(\gamma+k+1)} \tag{2.6}$$

for all $i(k)\in I$, where $i_0=1$, δ_i^j is the Kronecker delta. In [6], it was proved that if the parameters of the function (2.1) satisfy the inequalities $\gamma\ge\alpha\ge0$, $\gamma\ge\beta\ge0$, then BCF (2.2) converges to a finite value $f(\mathbf{z})$ for every $\mathbf{z}\in G$, where

$$G=\{\mathbf{z}\in\mathbb{C}^2:\ |z_1|\le h_1,\ |z_2|\le h_2\},$$

and h_1,h_2 are positive constants such that

$$8h_1(1+2h_2)(1-4h_1-h_2)+4(1+4h_1)h_2\le(1-4h_1-h_2)^2$$

and, in addition the convergent is uniformly on every compact subset of $\mathrm{Int}(G)$ to a function $f(\mathbf{z})$ holomorphic in $\mathrm{Int}(G)$. It was also proved that this BCF converges uniformly on every compact subset of the set

$$H=\bigcup_{\varphi\in(-\pi/2,\pi/2)} G_\varphi,$$

to a function $f(\mathbf{z})$ holomorphic in H, where

$$\begin{aligned}G_\varphi=\{\mathbf{z}\in\mathbb{C}^2:\ &\mathrm{Re}(z_1e^{-i\varphi})<\lambda_1\cos\varphi,\ |\mathrm{Re}(z_2e^{-i\varphi})|<\lambda_2\cos\varphi,\\ &|z_k|+\mathrm{Re}(z_ke^{-2i\varphi})<\nu_k\cos^2\varphi,\ k=1,2,\ |z_1z_2|-\mathrm{Re}(z_1z_2e^{-2\varphi})<\nu_3\cos^2\varphi\},\end{aligned}$$

where λ_1, λ_2, ν_1, ν_2, ν_3, μ_1, μ_2 are positive constants such that

$$\frac{\nu_2+4\nu_3}{\mu_2}\le\min\left\{2(1-\mu_1)-2\frac{\nu_1}{\mu_1},\ 2(1-4\lambda_1-\lambda_2-\mu_2)-\frac{2\nu_1+4\nu_3}{\mu_1}\right\}.$$

Note that for $\varphi=0$, the set H is the convergence set established in [3, Theorem 2]. Since the sets $\mathrm{Int}(G)$ and H contain a neighborhood of the origin, considering the proof of part (B) of [3, Theorem 2], we conclude that the function $f(\mathbf{z})$ is the analytic continuation of the function on the left side of equality (2.2) into $\mathrm{Int}(G)$ and H. Also, note that in both cases, no estimate for the rate of convergence of the BCF (2.2) was established.

Using expansion (2.2), let's construct the BCF expansion for the function

$$H_3(1,1;3/2;-\mathbf{z})=\frac{-1}{4\sqrt{z_2-z_1+z_2^2}}\ln\frac{1+2z_2-2\sqrt{z_2-z_1+z_2^2}}{1+2z_2+2\sqrt{z_2-z_1+z_2^2}}. \tag{2.7}$$

It is known [8] that

$$H_3(1,1;3/2;-\mathbf{z})=\sum_{r,s=0}^{+\infty}(-1)^{r+s}\frac{(1)_{2r+s}(1)_s}{(3/2)_{r+s}}\frac{z_1^rz_2^s}{r!s!}. \tag{2.8}$$

Setting $\alpha=0$, replacing γ with $\gamma-1$, and considering that $H_3(0,\beta,\gamma-1;\mathbf{z})=1$, from (2.2) we obtain

$$H_3(1,\beta;\gamma;\mathbf{z})=\cfrac{1}{1+\sum_{i_1=1}^{2}\cfrac{c_{i(1)}(\mathbf{z})}{d_{i(1)}(\mathbf{z})+\sum_{i_2=1}^{2}\cfrac{c_{i(2)}(\mathbf{z})}{d_{i(2)}(\mathbf{z})+\ddots}}}, \tag{2.9}$$

where the elements of the BCF are defined by (2.3)–(2.6), in which $\alpha=0$, and γ is replaced by $\gamma-1$. Next, setting $\beta=1,\gamma=3/2$ in (2.9), we obtain the BCF expansion for function (2.7):

$$H_3(1,1;3/2;-\mathbf{z})=\cfrac{1}{1+\sum_{i_1=1}^{2}\cfrac{c_{i(1)}(\mathbf{z})}{d_{i(1)}(\mathbf{z})+\sum_{i_2=1}^{2}\cfrac{c_{i(2)}(\mathbf{z})}{d_{i(2)}(\mathbf{z})+\ddots}}}, \tag{2.10}$$

where

$$c_1(\mathbf{z}) = \frac{4z_1}{3}, \quad c_2(\mathbf{z}) = \frac{2(1+4z_1)z_2}{3}, \tag{2.11}$$

$$d_{i(k)}(\mathbf{z}) = 1 + 2\frac{-2+\sum_{r=0}^{k-1}(\delta_{i_r}^1 - \delta_{i_r}^2)}{2k+1}\delta_{i_k}^2 z_2 + 4\frac{k+1+\sum_{r=0}^{k-1}\delta_{i_r}^2}{2k+1}\delta_{i_k}^2 z_1, \tag{2.12}$$

$$c_{i(k),1}(\mathbf{z}) = \frac{4(1+k+\sum_{r=0}^{k}\delta_{i_r}^2 + 2\delta_{i_k}^2 k z_2)\sum_{r=0}^{k}\delta_{i_r}^1 z_1}{(2k+1)(2k+3)}, \tag{2.13}$$

$$c_{i(k),2}(\mathbf{z}) = \frac{2(1+\sum_{r=0}^{k}\delta_{i_r}^2)(1+2\sum_{r=0}^{k}\delta_{i_r}^2)(1+4z_1)z_2}{(2k+1)(2k+3)} \tag{2.14}$$

for all $i(k) \in I$.

Since the parameters of the hypergeometric function in (2.8) satisfy the conditions $\gamma \geq \alpha \geq 0, \gamma \geq \beta \geq 0$, from the above we conclude that the BCF (2.10) converges to a finite value $f(\mathbf{z})$ for every $\mathbf{z} \in G$, and, moreover, converges uniformly on every compact subset of $\mathrm{Int}(G)$ to the function $f(\mathbf{z})$ holomorphic in $\mathrm{Int}(G)$, and also that (2.10) converges uniformly on every compact subset of H. Furthermore, the function $f(\mathbf{z})$ is the analytic continuation of function (2.7) into $\mathrm{Int}(G)$ and H.

Note that the problem of choosing optimal values for h_1, h_2 for G and $\lambda_1, \lambda_2, \nu_1, \nu_2, \nu_3, \mu_1, \mu_2$ for H remains open.

3. Approximation of the special function (2.7)

From a computational perspective, approximating H_3 using BCFs involves several key aspects: developing efficient algorithms that can compute approximants to H_3 with minimal computational resources while maintaining high accuracy; studying the convergence behavior of the BCF approximation, including the region of convergence and the rate of convergence; ensuring numerical stability in the recursive computation of approximants, so that rounding errors are not accumulated — which is especially important for high-order approximations.

Let $n \in \mathbb{N}$ be a fixed natural number and $f_n(\mathbf{z})$ be an nth approximant of the expansion (2.9).

For a comparative analysis of the accuracy and convergence rate of the function approximation, software was created, the main purpose of which is to calculate the approximate values of functions of two variables using two methods:

- Branched continued fraction: Calculation of the nth approximants according to the BCF expansion;
- Double power series: Calculation of the nth partial sums of the DPS.

Technology stack:

- Programming language: Python;
- Development environment: PyCharm.

Main components and functionality:

(A) BCF Computation
  - Classes implemented for calculating the partial numerators and denominators of the BCF.
  - A recursive approach using the backward recurrence algorithm is used to calculate the value of the nth approximant of the BCF, which involves calculating the tails of

the nth approximant according to the formulas:

$$G_{i(k)}^{(n)}(\mathbf{z}) = d_{i(k)}(\mathbf{z}) + \sum_{i_{k+1}=1}^{2} \frac{c_{i(k+1)}(\mathbf{z})}{G_{i(k+1)}^{(n)}(\mathbf{z})}, \quad i(k) \in I,\ 0 \le k \le n-1, \quad G_{i(n)}^{(n)} = d_{i(n)},\ i(n) \in I.$$

(B) Series Sum Computation
- Classes implemented for calculating the nth order partial sums of the DPS.
- Auxiliary methods implemented for calculating Pochhammer symbols, Kronecker deltas, and other related mathematical constructs.

(C) Analysis and Comparison

The software allows analysis and comparison of relative errors (Table 1) of the calculated function approximations at specified points $\mathbf{z}$ in the complex plane $\mathbb{C}^2$.

(D) Visualization

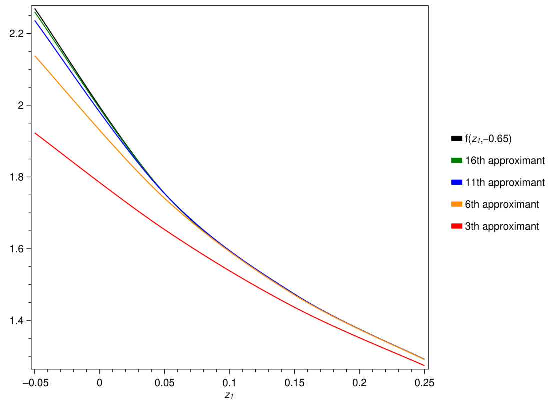
The software provides the ability to visualize results, compare approximation plots with the exact function values, and plot regions, where the BCF approximant and the power series partial sum achieve a specified approximation accuracy.

Let's analyze the approximation errors of function (2.7) using the BCF (2.10) and the DPS (2.8). Table 1 shows the relative errors of approximation at some points in $\mathbb{C}^2$ for function (2.7) by the 20th approximant $f_{20}(\mathbf{z})$ and the 20th partial sum $S_{20}(\mathbf{z})$. Analyzing the data in this table, we see that the BCF provides a better approximation than the DPS. Furthermore, there are points where the BCF converges, but the DPS diverges.

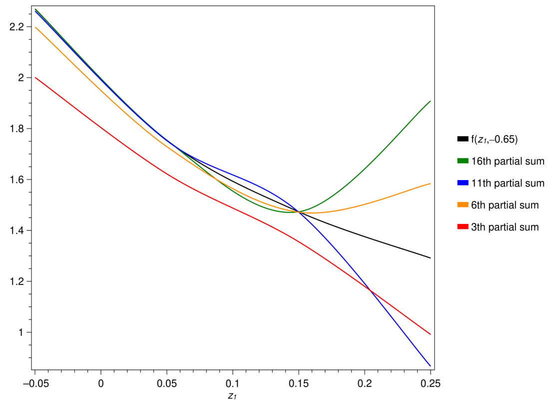
$\mathbf{z}$	**(2.10)**	**(2.8)**
(0.1250+0.0000i, 0.2500+0.0000i)	1.55×10^{-16}	5.07×10^{-09}
(0.1250+0.0000i, 0.3750+0.0000i)	6.88×10^{-14}	4.37×10^{-09}
(0.5000+0.0000i, 0.2500+0.0000i)	1.07×10^{-12}	$1.13\times10^{+04}$
(0.1083+0.0625i, 0.2165+0.1250i)	2.12×10^{-16}	5.39×10^{-09}
(0.3248+0.1875i, 0.4330+0.2500i)	1.37×10^{-13}	$2.60\times10^{+01}$
(0.4330+0.2500i, 0.6495+0.3750i)	5.61×10^{-12}	$6.77\times10^{+05}$
(0.0625+0.1083i, 0.1250+0.2165i)	3.14×10^{-16}	6.50×10^{-09}
(0.1875+0.3248i, 0.1250+0.2165i)	3.62×10^{-13}	$4.19\times10^{+01}$
(0.0000+0.1250i, 0.0000+0.2500i)	1.01×10^{-14}	9.05×10^{-09}
(0.0000+0.3750i, 0.0000+0.2500i)	9.20×10^{-12}	$6.01\times10^{+01}$
(-0.0625+0.1083i, -0.1250+0.2165i)	1.82×10^{-12}	1.48×10^{-08}
(-0.1875+0.3248i, -0.2500+0.4330i)	8.26×10^{-10}	$9.59\times10^{+01}$
(-0.0625-0.1083i, -0.1250-0.2165i)	1.82×10^{-12}	1.48×10^{-08}
(-0.1875-0.3248i, -0.2500-0.4330i)	8.26×10^{-10}	$9.59\times10^{+01}$
(0.0000-0.1250i, 0.0000-0.2500i)	1.01×10^{-14}	9.05×10^{-09}
(0.0000-0.3750i, 0.0000-0.2500i)	9.20×10^{-12}	$6.01\times10^{+01}$
(0.0625-0.1083i, 0.1250-0.2165i)	3.14×10^{-16}	6.50×10^{-09}
(0.1875-0.3248i, 0.1250-0.2165i)	3.62×10^{-13}	$4.19\times10^{+01}$
(1.0000+0.0000i, 2.0000+0.0000i)	9.73×10^{-09}	$3.71\times10^{+20}$

TABLE 1. Relative Errors of 20th Approximant and the 20th Partial Sum for (2.7)

Figure 1(A) and 1(B) show plots of the values of the BCF approximants (2.10), the partial sums of the DPS (2.8), and function (2.7) for $z_1 \in (-0.05, 0.25)$ and $z_2 = -0.65$. These figures also illustrate the advantages of approximating function (2.7) with the BCF compared to the DPS.



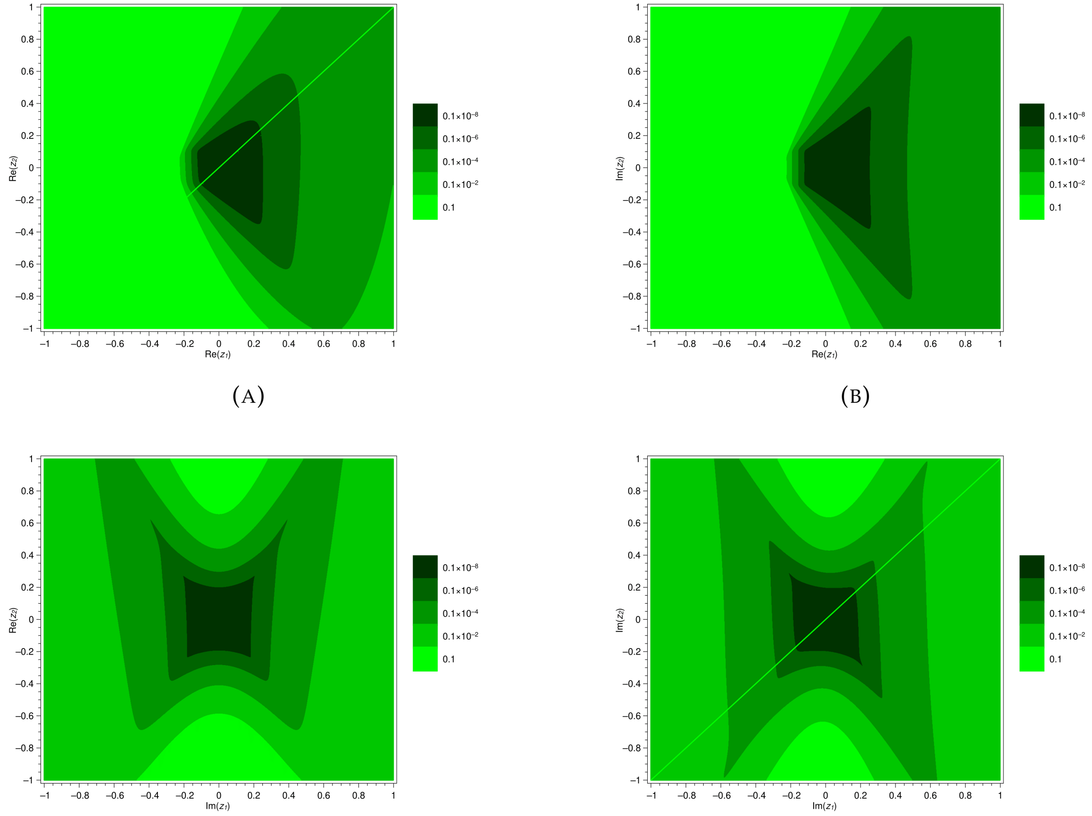
(A)



(B)

FIGURE 1. The plots of values of nth approximants of the BCF (A) and nth partial sums of the DPS (B) for function (2.7) with $z_2 = -0.65$.

Figure 2(A)–(D) show regions in different planes where the 11th approximant of the BCF (2.10) guarantees specified absolute error bounds for the approximation of function (2.7). Similarly, Figure 3(A)–(D) would depict regions where the 11th partial sum of the double power series (2.8) guarantees specified absolute error bounds for the approximation of function (2.7). As can be seen, the BCF (2.10) converges fastest in regions close to the origin, and the rate of convergence decreases as the distance from the origin increases. Considering the results of the applied research and the analytical expressions for the regions $\mathrm{Int}(G)$ and H, we conclude that there are wider convergence regions for the BCF (2.10), and consequently, wider regions of analytic continuation for the special function (2.7). Furthermore, the configurations of the regions in Figure 2(A)–(D) indicate the possibility of an analytical description of the convergence regions in different planes.



(A) (B) (C) (D)

FIGURE 2. Regions where the 11th approximant of BCF (2.10) guarantees specified absolute error bounds for the approximation of function (2.7) for (A) $\mathrm{Im}(z_1) = \mathrm{Im}(z_2) = 0$, (B) $\mathrm{Im}(z_1) = \mathrm{Re}(z_2) = 0$, (C) $\mathrm{Re}(z_1) = \mathrm{Im}(z_2) = 0$, (D) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2) = 0$.

4. COMPARATIVE ANALYSIS OF THE EFFICIENCY OF ALGORITHMS FOR COMPUTING APPROXIMANTS OF BRANCHED CONTINUED FRACTIONS

This section examines and compares four approaches to computing the nth approximant of the branched continued fraction (2.10): the classical backward recurrence algorithm (BR algorithm), its parallel implementations on CPU and GPU, as well as the continuant method. Particular attention is devoted to the execution speed of the algorithms and their robustness against the accumulation of rounding errors.

We consider the approximant of the branched continued fraction as a tree-like structure of depth n, whose nodes are the coefficients $c_{i(k)}$ and $d_{i(k)}$. Each node of the tree is characterized by a multiindex `index` = $(i_1, i_2, \ldots, i_k)$, where $k-1$ denotes the depth level (the layer of the branched continued fraction) for coefficients $c_{i(k)}$ and k denotes the depth level for coefficients $d_{i(k)}$.

The classical BR algorithm computes the nth approximant of the branched continued fraction by performing a reverse traversal (bottom-up), that is, starting from the leaf coefficients

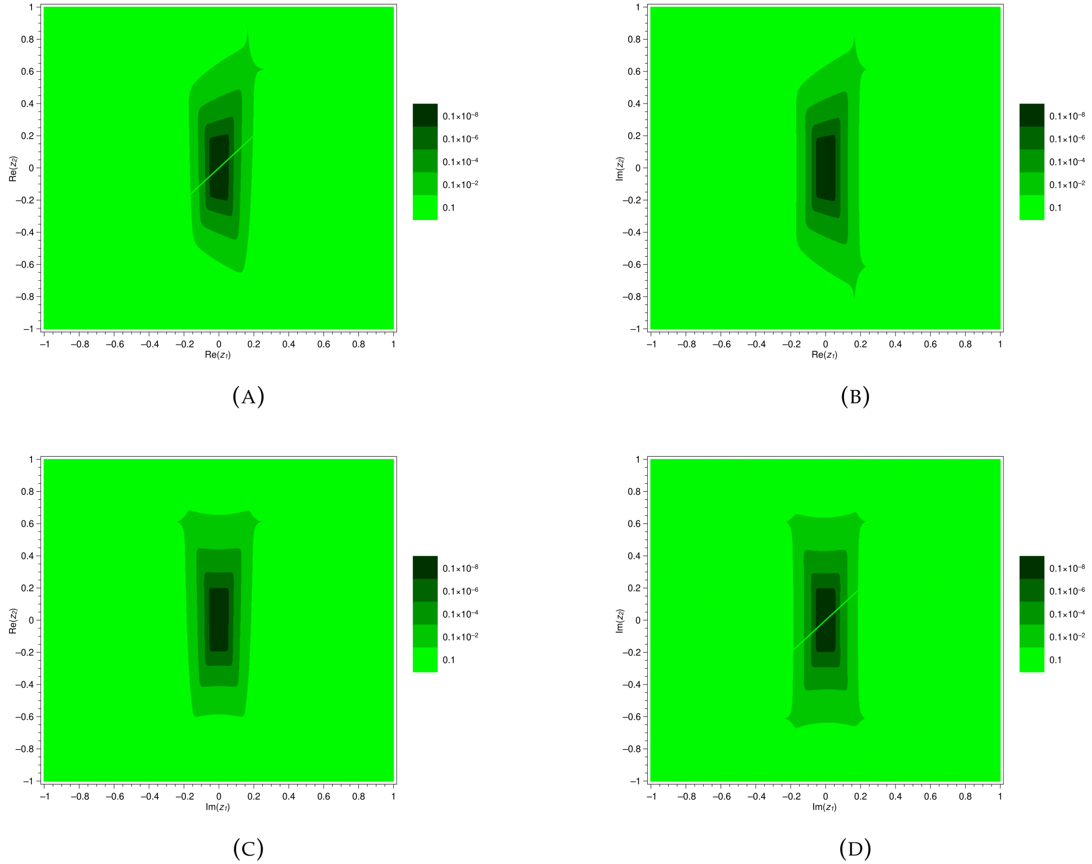


FIGURE 3. Regions where the 11th partial sum of power series (2.8) guarantees specified absolute error bounds for the approximation of function (2.7) for (A) $\mathrm{Im}(z_1) = \mathrm{Im}(z_2) = 0$, (B) $\mathrm{Im}(z_1) = \mathrm{Re}(z_2) = 0$, (C) $\mathrm{Re}(z_1) = \mathrm{Im}(z_2) = 0$, (D) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2) = 0$.

and proceeding towards the root of the fraction. The algorithm is implemented as a recursive function that takes three arguments:

1. `depth` — the current recursion depth (with value 0 corresponding to the coefficients d_0, c_1 and c_2).
2. `index` — an array of length n representing a multi-index that uniquely identifies the path to a coefficient within the fraction structure.
3. `n` — the order of the approximant, corresponding to the maximum depth of computation.

To access the coefficients of the fraction, auxiliary functions `get_d(index, depth)` and `get_c(index, depth)` are employed, which return the values $d_{i(k)}$ and $c_{i(k)}$, respectively.

The pseudocode of the BR algorithm has the following form:

Algorithm 1 BR-algorithm

```
function BR_RECURSIVE(index, depth, n)
    v ← get_d(index, depth)                         ▷ Retrieve coefficient d at current depth
    if depth == n then
        return v                                    ▷ Maximum depth reached
    end if
    for i = 1 to 2 do                               ▷ Sum contributions of child branches
        index[depth] ← i
        v ← v + get_c(index, depth)/br_recursive(index, depth + 1, n)
    end for
    return v
end function
```

This implementation allows one to compute the value of the branched continued fraction by invoking the function with arguments `depth = 0` and an initial array `initial_index = (0, 0, ..., 0)` of length n. Moreover, it makes it possible to compute any remainder of the approximant (a subfraction) by specifying the required multi-index and the depth of the first coefficient $d_{i(k)}$ of this subfraction.

Here we consider a parallel version of the BR algorithm in which multiple processes compute subfractions simultaneously on the CPU. At each depth level, these subfractions can be calculated independently, as they do not depend on one another until their results are summed. This allows the workload to be distributed across several processes, speeding up the execution of the algorithm.

The parallel BR algorithm can be formally described by the following pseudocode:

A parallel version of the BR algorithm has also been implemented using the Numba library, which enables the creation of CUDA [17] kernels for computations on the GPU. This implementation is similar to the CPU version; however, GPU threads are used instead of processes. Since Numba does not support recursive calls within CUDA kernels, the recursive function was replaced with an iterative implementation using a loop and a stack to simulate recursion.

The continuant method computes the value of the nth approximant of the branched continued fraction as the ratio of the determinants of two matrices, C_0 and C_1. The matrix C_0 is constructed such that its main diagonal contains the coefficients $d_{i(k)}$, the entries above the main diagonal are the coefficients $c_{i(k)}$, and symmetrically to them (below the main diagonal) are the values -1; all other elements of the matrix are zero. The matrix C_1 is obtained from C_0 by removing the first row and the first column.

The form of the matrix C_1 is presented below:

$$\begin{pmatrix} d_0 & c_1 & c_2 & 0 & 0 & 0 & 0 & \cdots \\ -1 & d_1 & 0 & c_{11} & c_{12} & 0 & 0 & \cdots \\ -1 & 0 & d_2 & 0 & 0 & c_{21} & c_{22} & \cdots \\ 0 & -1 & 0 & d_{11} & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & d_{12} & 0 & 0 & \cdots \\ 0 & 0 & -1 & 0 & 0 & d_{21} & 0 & \cdots \\ 0 & 0 & -1 & 0 & 0 & 0 & d_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The continuant method can be formally described by the following pseudocode:

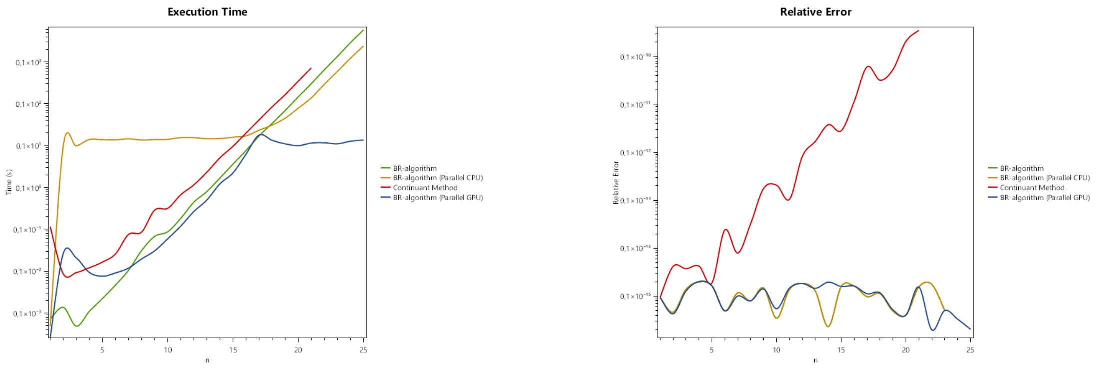
Algorithm 2 Parallel BR-algorithm

function RECURSIVE_PARALLEL(n, num_processes)
  if $n == 1$ **then**
    return BR_RECURSIVE(initial_index(n), 0, n)
  end if
  ▷ Determine the optimal depth for parallelization
  $parallel\_depth \leftarrow \lfloor \log_2(num\_processes) \rfloor$
  if $parallel\_depth \geq n$ **then**
    $parallel\_depth \leftarrow n - 1$
  end if
  ▷ Generate the set of all multi-indices of length $parallel\_depth$ in lexicographic order
  $indices\_set \leftarrow$ generate_indices(length=parallel_depth)
  ▷ 1. PARALLEL PHASE: compute independent subfractions
  $results \leftarrow \{\}$ ▷ Storage for results
  for all $idx \in indices\_set$ **in parallel do**
    $results[idx] \leftarrow$ BR_RECURSIVE$(idx, parallel_depth, n)$
  end for
  ▷ 2. SEQUENTIAL PHASE: compute the upper part of the fraction
  ▷ Modify `get_d` to return precomputed results at $parallel\_depth$
  function GET_D_MEMOIZED(index, depth)
    if $depth == parallel\_depth$ **then**
      return $results[index]$
    else
      return GET_D_ORIGINAL(index, depth)
    end if
  end function
  return BR_RECURSIVE(initial_index(n), 0, parallel_depth)
end function

Algorithm 3 Continuant Method

function CONTINUANT_METHOD(n)
  ▷ Construct the matrix C_0 according to the fraction structure
  $C_0 \leftarrow$ construct_matrix(n)
  ▷ Obtain the matrix C_1
  $C_1 \leftarrow C_0$ without the first row and column
  ▷ Perform LU decomposition for both matrices
  $L_0, U_0 \leftarrow$ LU_decomposition(C_0)
  $L_1, U_1 \leftarrow$ LU_decomposition(C_1)
  ▷ Extract diagonal elements of U
  $d_0 \leftarrow$ diagonal elements of U_0
  $d_1 \leftarrow$ diagonal elements of U_1
  ▷ Note: computing the determinants directly may cause variable overflow, so we use element-wise division of the diagonal elements instead
  ▷ Calculate the result as the product of diagonal elements ratios
  return $d_0[0] \cdot \prod_{i=1}^{\text{length}(d_0)-1} \frac{d_0[i]}{d_1[i-1]}$
end function

Figure 4 shows that the relative error of computing approximants using the continuant method increases with n, whereas the relative error for different implementations of the BR algorithm remains within machine epsilon. Furthermore, the continuant method requires a significant amount of memory, as the sizes of the matrices C_0 and C_1 grow exponentially with n. In this study, approximants were computed only up to $n = 21$, since for $n = 22$ the required memory would exceed 32 GB.



(A) (B)

FIGURE 4. Execution time and relative error of computing the branched continued fraction approximant (2.10).

For small values of n (up to 7), the classical BR algorithm implementation is the fastest. For $n > 7$, the GPU implementation of the BR algorithm becomes faster than the classical one. In all cases, the continuant method is slower than the classical BR algorithm and, for $n > 18$, is outperformed by the parallel BR algorithm on the CPU.

The BR algorithm executed in parallel on the CPU involves process creation, which requires additional time, making it slower than other implementations for $n \leq 18$ (see Fig. 5). However, for $n > 18$, the advantage of parallel execution outweighs the overhead of process creation, and this implementation becomes faster than both the classical BR algorithm and the continuant method.

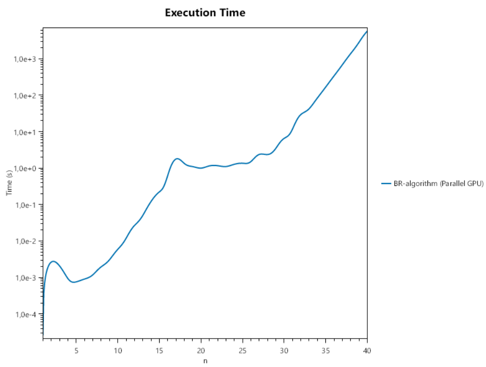


FIGURE 5. Execution time of the parallel GPU implementation of the BR algorithm.

As the value of n increases, the parallel GPU implementation of the BR algorithm engages more threads, but only at the last level of the fraction. This means that the workload per individual GPU thread does not increase; instead, the number of active threads and the load on the CPU grow. For $n > 16$, however, the additional workload falls solely on the GPU, while the CPU workload remains constant. Consequently, in the range $17 \leq n \leq 25$, the growth of execution time slows down: all GPU threads are fully utilized, and each thread handles a relatively small portion of the work. For $n > 25$, the workload per thread increases significantly again, leading to a faster increase in execution time.

5. Conclusion

In this work, using the expansion of the ratio of Horn's hypergeometric functions H_3 [5], an expansion for the special function (2.7) into BCF (2.10) was constructed. To construct and analyze the approximations (using nth approximants and nth partial sums) of this special function, specialized software was developed. The analysis of the applied research showed that the nth approximants converge fastest in regions close to the origin, and the rate of convergence decreases with increasing distance from the origin. Furthermore, the constructed BCF expansion has a wider convergence region and a better rate of convergence compared to the corresponding hypergeometric series.

Numerical experiments confirmed the feasibility and effectiveness of using BCFs as a tool for approximating special functions, particularly hypergeometric functions. Graphical illustrations indicate the potential for developing a new and promising direction in the analytical theory of continued fractions: the study of convergence regions of BCFs in various planes. The practical results obtained can also be used to investigate the convergence of BCF expansions of other special functions.

Additionally, different implementations of the BR algorithm and the continuant method were analyzed in terms of execution time and rounding errors. The BR algorithm was found to be stable, whereas the continuant method exhibited instability. Moreover, the continuant method is slower than the other approaches. The parallel implementation of the BR algorithm is faster than the single-threaded version, and its performance improves as more threads are used.

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Research Article

Mathematical modelling of control-loss detection via risk-sensitive reinforcement learning on partially observable Markov decision processes

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ABSTRACT. This work introduces a method for identifying high-risk loss-of-control episodes in digital settings by combining risk-sensitive reinforcement learning with decision-making under partial observability. We motivate the need to reason with incomplete and noisy information — typical of real-world deployments and, in particular, of monitoring user behaviour during critical states. The agent-environment interaction is modelled within the partially observable Markov decision process formalism, which maintains a belief (probabilistic posterior) over latent states given histories of actions and observations. Behaviour is analysed at the trajectory level, and tail risk is quantified via the Conditional Value at Risk (CVaR), enabling the assessment of expected losses in worst-case regimes rather than average-case performance. To ensure transparency and foster trust, we integrate explainable AI (XAI) techniques that reveal the factors driving risk estimates and action choices. The resulting pipeline provides a principled basis for adaptive detection of critical states and for early-warning interventions in complex digital environments, supporting reliable and accountable decision support.

Keywords: Reinforcement learning, partially observable Markov decision process, risk-sensitive metric, explainable AI techniques.

2020 Mathematics Subject Classification: 90C40, 93C41, 68T05.

1. INTRODUCTION

In rapidly evolving sociotechnical systems, where signals are noisy, structure is complex, and uncertainty is pervasive, there is a growing need to identify hazardous or critical modes in a timely manner. Central to this agenda is the notion of loss of control: episodes in which a human operator or an automated system deviates from intended behavior under incomplete knowledge of the true environmental state. Such episodes can precipitate severe outcomes – ranging from user dependencies and anomalous activity to security-policy violations and high-stakes decision errors.

Models that presuppose full state observability – e.g., classical Markov Decision Processes (MDPs) – are ill-suited to these settings, which are inherently stochastic and only partially observed. A more faithful representation is provided by the Partially Observable Markov Decision Process (POMDP), which maintains a probabilistic belief over latent states updated from the history of actions and observations, thereby offering a principled mechanism for operating under deep uncertainty.

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A further challenge is explicit treatment of downside risk in atypical or adversarial regimes. Risk-sensitive criteria such as Conditional Value at Risk (CVaR) quantify losses in the distributional tail and, unlike expectation-centric objectives, emphasize the magnitude of adverse outcomes. This focus supports the synthesis of robust, risk-aware policies.

Coupling reinforcement learning with the POMDP formalism yields an adaptive toolkit: the agent incrementally incorporates new evidence, updates its policy from experience, and responds to environmental drift in real time. To promote trust and transparency, explainable AI (XAI) techniques can be used to interpret internal decision pathways and improve model interpretability.

Against this backdrop, the present work introduces a risk-sensitive framework for detecting loss-of-control episodes that integrates POMDP modelling, CVaR-based objectives, and reinforcement learning. The resulting approach enables early warning and identification of critical situations under partial observability and high uncertainty.

2. Illustrative scenarios of control loss

To illustrate the practical scope of the approach, we outline scenarios in which an individual undergoes a transient loss of control – understood here as a behavioral episode that yields direct or indirect adverse outcomes and is later recognized by the individual or their environment as erroneous.

First, consider social-media use. Users may impulsively post messages or comments that are subsequently judged inappropriate or reputation-damaging. Beyond basic content filters, organizations could adopt configurable, policy-aware pre-publication checks – such as a browser extension – that warn when draft content conflicts with internal guidelines. This is especially pertinent for official or corporate accounts.

A second setting is online gambling. During affective or impulsive states, users may assume excessive risk, with substantial financial consequences. As regulation increasingly mandates responsible-gambling safeguards, a natural extension is to integrate predictive modules that detect early indicators of risk-seeking behavior and trigger proportionate interventions (e.g., recommending a break or notifying a pre-designated contact). Selective application to vulnerable cohorts can align user protection with legal and ethical expectations.

A third scenario pertains to software deployment in CI/CD pipelines. Under deadline pressure, engineers may bypass tests or override safeguards, pushing misconfigured builds to production with downstream organizational impact. Predictive modules can monitor contextual and behavioral indicators (e.g., recent failure history, code churn, out-of-hours commits) and trigger proportionate interventions – requesting peer approval, recommending a canary rollout, or deferring deployment to a lower-risk window. These mechanisms operationalize loss-of-control detection in high-stakes technical workflows and align with existing DevOps governance.

These examples ground the abstract notion of control loss in concrete contexts and motivate the value of the proposed modelling framework.

3. The agent as a Markov decision process

The Markov Decision Process (MDP) [2, 9, 15] is adopted as the primary abstraction for agent–environment interaction because it offers a mathematically tractable way to model tasks in which actions drive stochastic state transitions and immediate outcomes in a dynamic setting. In effect, the MDP encodes sequential decision-making and the causal dependencies between states, actions, and rewards. Formally, an MDP is a tuple

$$\mathcal{M} = \langle S, A, \tau, r, \gamma \rangle,$$

where $\tau(s'|s,a)$ is the transition kernel, $r(s,a) \in \mathbb{R}$ is the one-step reward, and $\gamma \in [0,1]$ is the discount. For any policy $\pi(a|s)$, the value functions satisfy the Bellman relations:

$$V^\pi(s) = \sum_a \pi(a|s) \left[r(s,a) + \gamma \sum_{s'} \tau(s'|s,a) V^\pi(s') \right], Q^\pi(s,a) = r(s,a) + \gamma \sum_{s'} \tau(s'|s,a) \sum_{a'} \pi(a'|s') Q^\pi(s$$

Moreover, the MDP formalism underpins modern Reinforcement Learning (RL) algorithms and theory [18, 19, 20].

A standard MDP presumes full observability: at each time t, the agent observes the true state s_t prior to choosing a_t. In problems concerned with detecting moments of loss of control, this assumption is rarely justified. The monitoring system does not access latent internal conditions (e.g., cognitive or emotional states) and instead receives only indirect, noisy signals. Let z_t denote the observed proxy vector (e.g., reaction time, message or bet size, click frequency, keystroke error rate); decisions must then be made under partial observability.

To explicitly encode uncertainty about the hidden state, we model the problem as a partially observable MDP (POMDP) [11, 13, 17, 18]. The agent maintains a belief state b_t, a probability distribution over latent states representing its posterior information after past actions and observations [4, 16]. Given action a_t and new observation z_{t+1}, the belief is updated by the Bayes filter:

$$b_{t+1}(s') = \eta \, o(z_{t+1}|s', a_t) \sum_{s \in S} \tau(s'|s, a_t) b_t(s), \tag{3.1}$$

with

$$\eta^{-1} = \sum_{s'} o(z_{t+1}|s', a_t) \sum_{s} \tau(s'|s, a_t) b_t(s). \tag{3.2}$$

Planning then proceeds over beliefs, enabling policies that act because of uncertainty rather than in spite of it; empirical performance typically improves in realistic, noisy environments.

From a formal perspective, passing to the belief space $\mathcal{B} = \Delta(\mathcal{S})$ converts the partially observable control problem into a fully observable (generally continuous) belief-MDP. The one–step belief reward and the (stochastic) belief transition induced by the Bayes operator Φ are

$$R(b,a) = \sum_s b(s) R(s,a), \quad b' = \Phi(b,a,z), \tag{3.3}$$

so that the optimal value function solves

$$V^*(b) = \max_{a \in A} \left\{ R(b,a) + \gamma \mathbb{E}_{z \sim p(\cdot|b,a)} \left[V^*(\Phi(b,a,z)) \right] \right\}, \tag{3.4}$$

with $p(z|b,a) = \sum_{s'} o(z|s',a) \sum_s \tau(s'|s,a) b(s)$.

A POMDP can be specified by the tuple

$$\mathcal{P} = \langle S, A, Z, \tau, o, r, \gamma \rangle, \tag{3.5}$$

S – latent state space (true configurations are not directly observable);
A – action space (available controls at each step);

Z – observation space (noisy signals emitted by the system);
$\tau(s'|s,a)$ – state transition kernel;
$o(z|s',a)$ – observation kernel (likelihood of z after arriving in s' via a);
$r(s,a) \in \mathbb{R}$ – immediate reward.

Intuitively, τ captures uncertainty in the underlying dynamics, o connects latent states to observable evidence, and r quantifies instantaneous utility. Decisions are then computed over beliefs $b \in \mathcal{B}$, leveraging the updates above to integrate new information and act optimally under partial information.

The formalism of a partially observable Markov decision process offers a versatile framework for modelling decision-making under uncertainty. To situate its potential, it is helpful to consider uses in neighbouring domains. A broad survey of applications is provided in [3], although the particular use case studied here is not treated explicitly. Structurally closest analogues arise in behavioural ecology (scientific domain), in medical diagnostics (social domain) and in corporate policy (business domain).

In behavioural ecology, a common premise is that an organism behaves (approximately) optimally with respect to an internal representation of its environment. The analyst posits candidate states, actions, observations, and rewards thought to be salient for the organism, derives an optimal strategy from this specification, and compares it with observed behaviour. Mismatches motivate alternative hypotheses and iterative refinement of the model, progressively deepening our understanding of behaviour–environment interactions.

Our focus differs: rather than incrementally perfecting the latent model, we seek to detect marked departures from the putative optimal strategy. Such departures are interpreted as indicators of a loss of self-control in the individual under observation.

In medical diagnostics, despite substantial advances, practice still hinges on expert judgement and remains error-prone. The core difficulty is that the patient's internal state is only partially observable. The clinician must choose among actions–laboratory tests, medication, surgery, physical therapy–each carrying costs and risks, yet providing only partial information about the underlying condition. Consequently, one must balance observational accuracy against financial or medical burden.

In a context relevant to our work, where gambling addiction is formally recognised as a disorder [21, 22], the model's agent can be cast as a virtual "physician" tasked with inferring loss of control from limited, indirect observations. The objective is not merely to explain behaviour post hoc, but to identify early signals of deterioration in a principled, uncertainty-aware manner.

In corporate governance, a strand of POMDP-based work treats organizational control problems – internal audits, accounting, and policy enforcement – as sequential decision making under partial observability [3]. The key premise is system-level: while individual agents (employees) perceive only fragments, an internal model can aggregate telemetry from across the enterprise to infer latent organizational states more reliably than any single decision-maker. This separation between human-limited views and a model's integrated perspective motivates the use of belief-state monitoring for stability and compliance at scale.

Our setting instantiates this perspective in software delivery pipelines. Deployment readiness is only indirectly observable through proxies such as test coverage, recent failure history, code churn, review latency, or out-of-hours activity; actions (e.g., triggering additional tests, requesting peer approval, switching to a canary rollout, or deferring release) trade information gain and risk reduction against cost and delay. Modeling the pipeline as a POMDP allows principled balancing of these trade-offs, and provides a mechanism for detecting loss-of-control episodes under deadline pressure – when safeguards are bypassed or misconfigurations propagate – by elevating intervention intensity as the belief shifts toward hazardous latent states.

4. Reinforcement Learning as a tool for detecting control loss risk

The central premise of applying reinforcement learning to behavioral modelling is to specify an agent that acquires an optimal policy by interacting with its environment. Formally, this interaction is represented as a sequence of states, actions, and rewards, consistent with the structure of a Markov Decision Process (MDP) or, under partial observability, its POMDP extension.

The goal of the agent is to find a policy $\pi(a|s)$ that maximizes the expected cumulative reward:

$$\pi^* = \arg\max_{\pi} \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R(s_t, a_t)|\pi\right], \tag{4.6}$$

where $\gamma \in [0, 1]$ is the discount factor that governs the relative weight of future returns.

In behavioral analytics, attention shifts from isolated actions or momentary states to the evaluation of the complete behavioral trajectory:

$$\tau = (s_0, a_0, r_0, s_1, a_1, r_1, \ldots, s_T), \tag{4.7}$$

which aggregates the downstream effects of the agent's decisions. This trajectory-level view reveals persistent action patterns that can culminate in adverse outcomes. Rather than flagging single "risky" moves, the analysis considers how conditions accumulate over time and gradually elevate the likelihood of control loss.

Within this perspective, it is natural to specify a subset of critical states $s_{crit} \subseteq S$ in which the probability of loss of control (e.g., progression to anomalous or addictive behavior) rises markedly. Behavior is then interpreted not merely as a reaction to the present state but as the cumulative consequence of prior choices. Consequently, a risk-detection mechanism should avoid judging the current action in isolation and instead condition its assessment on the full historical context encoded in τ.

To assess risk in dynamic or stochastic decision-making, we employ a trajectory-level risk-sensitive criterion. This perspective goes beyond average performance to capture the probability mass of rare but consequential failures, thereby measuring the system's vulnerability to adverse, off-nominal conditions that fall outside typical expectations.

Within this setting, the Conditional Value at Risk (CVaR) [14, 23] at confidence level α, denoted $CVaR_\alpha$, is appropriate. CVaR is the expected loss conditional on exceeding a quantile threshold defined by the Value at Risk (VaR) [5, 10, 12]. Equivalently, $CVaR_\alpha$ computes the mean loss over the worst $100 \cdot (1-\alpha)\%$ of outcomes – the tail of the loss distribution:

$$CVaR_\alpha = \mathbb{E}\left[L|L \geq VaR_\alpha\right], \tag{4.8}$$

where L is a loss-valued random variable. In reinforcement learning, L can be instantiated as the cumulative negative reward incurred along a trajectory, interpreting negative reward as a penalty for unsafe or undesirable behavior that degrades policy performance [18].

Unlike VaR – which merely identifies the α-quantile loss not to be exceeded with probability α – CVaR quantifies tail severity, i.e., the expected magnitude of loss once the system has entered worst-case regimes.

While we adopt $CVaR_\alpha$ as the primary risk functional due to its coherence, tail sensitivity, and convexity – properties that facilitate stable optimization under partial observability – it is not the only admissible choice. Alternatives such as entropic (exponential-utility) risk, mean–variance criteria, and distributional/quantile losses may be preferable in domain-specific deployments. Our pipeline is risk-functional agnostic: $CVaR_\alpha$ can be replaced by other coherent or utility-based measures without modifying the overall design.

For reinforcement learning or trajectory-level analysis, this specialization takes the form

$$CVaR_\alpha\left(-\sum_{t=0}^{T} r_t\right), \tag{4.9}$$

where r_t denotes the immediate reward at time t and the sum spans the entire trajectory. The leading minus sign enforces the standard loss-as-negative-reward convention used in risk-sensitive optimization. Hence, the metric captures the average loss among trajectories with the lowest (worst) cumulative returns.

We define a trajectory-level risk estimator as

$$\mathcal{R}(\tau) : \tau \mapsto [0, 1], \tag{4.10}$$

using data from offline reinforcement learning (offline RL) [7, 8], is trained to discriminate between trajectories that are safe and those that are risky. This enables early warning of potentially hazardous behavioral patterns and supports risk-aware sorting of trajectories, with an emphasis on highlighting critical states.

Because offline RL relies on a fixed logged dataset, the risk estimator is vulnerable to distributional shift between training data and deployment, which can degrade reliability. In this study we restrict interpretation to in-distribution regimes (i.e., behaviour supported by the logged data) and note that out-of-support trajectories require additional safeguards and validation in real-world evaluations.

By computing risk-sensitive metrics at the trajectory level and selecting appropriate decision thresholds, the system can flag episodes in which the likelihood of control loss becomes non-negligible. A complementary ingredient of the approach is explainable AI (XAI) [1, 6], which is essential for making the RL model's behavior transparent and for establishing trust in its outputs – especially in applications involving anomalous usage or emergent behavioral addictions, where model decisions may directly affect a person's psycho-emotional well-being.

Missed detections (false negatives) are particularly consequential: failing to recognize a loss-of-control event can lead to substantial harm. Moreover, the very presence of a monitoring system can induce a perception of external safety, potentially attenuating users' internal safeguards; if the model then fails to intervene when needed, it not only forfeits its protective role but may amplify the resulting harm due to misplaced reliance.

Symmetrically, false positives carry significant costs for organizations deploying such systems. Over-triggering on benign behavior can frustrate users, reduce engagement, and erode trust – ultimately undermining competitiveness and complicating real-world adoption.

For these reasons, XAI plays a central role: it allows developers, users, and independent auditors to understand why a particular trajectory was flagged as risky or anomalous, improving decisional transparency and enabling calibration to application-specific requirements for sensitivity and specificity. Further gains in interpretability can be obtained with attention-based mechanisms that automatically surface the most informative segments of the input – across time or feature dimensions. Visualizing attention weights (e.g., as heatmaps) clarifies which portions of the trajectory or which state features most influenced the risk estimate, providing deeper insight into the agent's decision logic while remaining consistent with the overall model structure.

5. Conclusion

This work examined a risk-sensitive framework for detecting loss-of-control episodes in digital settings by integrating partially observable models (POMDP), reinforcement learning, and risk-aware objectives. We argued that the POMDP formalism is well suited to decision-making under uncertainty typical of real-world systems in which full state observability is infeasible.

Special emphasis was placed on Conditional Value at Risk (CVaR) as a trajectory-level criterion for downside risk. By quantifying tail losses in worst-case regimes, CVaR introduces a principled layer of caution, steering policies not only toward maximizing expected returns but also toward avoiding low-probability, high-impact failures.

We also justified the use of explainable AI (XAI) to expose internal decision pathways, clarify causal relations, and strengthen trust among end users and operators.

Taken together, the proposed concept supports the development of adaptive intelligent systems capable of: detecting critical shifts in behavioral patterns, providing explanations of decisions, synthesizing control strategies that remain robust under uncertainty and rare but destructive events.

Using both online and offline reinforcement learning yields a flexible foundation for real-time systems that adapt to environmental changes, accumulate experience, reassess risk profiles, and preempt critical outcomes.

While the proposed framework shows promise under controlled conditions, it has not yet been validated on large-scale, real-world datasets; consequently, its practical robustness remains unproven. We therefore interpret the present results as feasibility evidence rather than deployment-ready performance, particularly in the presence of distribution shift and operational constraints.

In future work, we plan to instantiate the framework in concrete domains, such as detecting critical behavioral states on online platforms and analyzing autonomous agents operating under severe uncertainty. We will also pursue advanced XAI mechanisms that provide multistep explanations for sequential decisions. Finally, data-driven structural and parametric identification of POMDP models from empirical evidence – beyond the scope of this study – will be addressed to enable practical deployment and validation.

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Research Article

Truncation error bounds of branched continued fraction expansions of special ratios of Horn's hypergeometric functions H_4

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ABSTRACT. The paper considers the branched continued fraction extensions of special ratios of Horn's hypergeometric functions H_4 with real parameters and variables. Truncation error bounds are established for such expansions with certain conditions on their coefficients. Some domains of analytical continuation of the above-mentioned special ratios are also established using the PF method (based on the so-called property of fork for approximants of a branched continued fraction).

Keywords: Hypergeometric function, branched continued fraction, approximation by rational functions, rate of convergence, analytic continuation.

2020 Mathematics Subject Classification: 33C65, 32A17, 32A10, 41A25, 30B40.

1. INTRODUCTION

The study of special functions has been and remains relevant for several centuries due to their practical application in many fields of science [6, 16, 20, 22].

The paper investigates branched continued fraction expansions of the ratios of hypergeometric functions of two variables [15, 16, 17]. An overview of such expansions was described in [3]. The application of branched continued fraction expansions to the approximation of special functions represented by double hypergeometric series were considered in [1, 2, 7, 14]. In this paper, we continue our study of the expansions of special ratios of the Horn's hypergeometric functions H_4 [2, 9, 10, 11, 12, 8].

Recall that the function H_4 is defined as (see [15, Section 5.7] and [17])

$$H_4(\alpha,\beta;\gamma,\delta;\mathbf{z})=\sum_{r,s=0}^{\infty}\frac{(\alpha)_{2r+s}(\beta)_s}{(\gamma)_r(\delta)_s}\frac{z_1^r}{r!}\frac{z_2^s}{s!},$$

where $\alpha,\beta,\gamma,\delta\in\mathbb{C}$ herewith $\gamma,\delta\notin\{0,-1,-2,\dots\}$, $\mathbf{z}=(z_1,z_2)\in\mathfrak{D}_{p,q}$,

$$\mathfrak{D}_{p,q}=\left\{\mathbf{z}\in\mathbb{C}^2:\ |z_1|<p,\ |z_2|<q\right\},\ p>0,\ q>0,\ 4p=(q-1)^2,\ q\neq1, \tag{1.1}$$

$(\xi)_k$ is the Pochhammer symbol, $(\xi)_k=\Gamma(\xi+k)/\Gamma(\xi)$, $\Gamma(z)$ is the gamma function.

Note that some relations for the Horn's hypergeometric function H_4 were obtained in [5, 19, 21], including differentiation and integration formulas, series for special values of parameters and variables, and some generating functions for various special functions in terms of this function.

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As a special case of [1, Theorem 1], we have the following theorem:

Theorem A. *The ratios*

$$\frac{H_4(\alpha,\beta;\gamma,\beta;\mathbf{z})}{H_4(\alpha+1,\beta;\gamma+1,\beta;\mathbf{z})}, \tag{1.2}$$

$$\frac{H_4(\alpha,\delta+1;\gamma,\delta;\mathbf{z})}{H_4(\alpha+1,\delta+1;\gamma,\delta+1;\mathbf{z})}, \tag{1.3}$$

and

$$\frac{H_4(\alpha,\delta+1;\gamma,\delta;\mathbf{z})}{H_4(\alpha,\delta+2;\gamma,\delta+1;\mathbf{z})} \tag{1.4}$$

have formal branched continued fraction expansions

$$1-z_2-\cfrac{a_1z_1}{1-z_2-\cfrac{a_2z_1}{1-z_2-\cfrac{a_3z_1}{1-\ddots}}}, \tag{1.5}$$

where

$$a_k=\frac{(2\gamma-\alpha+k-1)(\alpha+k)}{(\gamma+k-1)(\gamma+k)},\ k\ge 1,$$

$$1-\frac{\delta-\alpha}{\delta}z_2-\cfrac{b_1z_1}{1-z_2-\cfrac{b_2z_1}{1-z_2-\cfrac{b_3z_1}{1-\ddots}}}, \tag{1.6}$$

where

$$b_1=\frac{2(\alpha+1)}{\gamma},\ b_k=\frac{(2\gamma-\alpha+k-3)(\alpha+k)}{(\gamma+k-2)(\gamma+k-1)},\ k\ge 2, \tag{1.7}$$

and

$$1+\cfrac{d_0z_2}{1-d_1z_2-\cfrac{c_1z_1}{1-d_2z_2-\cfrac{c_2z_1}{1-\ddots}}}, \tag{1.8}$$

where

$$\begin{aligned} d_0&=\frac{\alpha}{\delta(\delta+1)},\ d_1=1-\frac{\alpha}{\delta+1},\ c_1=\frac{2(\alpha+1)}{\gamma},\\ d_k&=1,\ c_k=\frac{(2\gamma-\alpha+k-3)(\alpha+k)}{(\gamma+k-2)(\gamma+k-1)},\ k\ge 2, \end{aligned} \tag{1.9}$$

respectively.

Different domains of analytical continuation of ratio (1.2) by branched continued fraction (1.5) were established in [1, 12] by the PC method (based on the so-called principle of correspondence between a formal double power series and a branched continued fraction). Some domains of convergence of the expansion (1.6) and (1.8) was studied in [11] and [10], respectively. Truncation error bound for expansion (1.5) with certain conditions on real parameters was established in [8]. Here, a new domain of analytical continuation of (1.2) was also established by the PF method (based on the so-called property of fork for approximants of a branched continued fraction).

As in the theory of continued fractions [18], we have the following definition:

Definition A. *Let for each* $\mathbf{z}\in\mathfrak{D}$, $\mathfrak{D}\subset\mathbb{C}^2$, *the branched continued fraction*

$$v_0(\mathbf{z})+\sum_{i_1=1}^{2}\cfrac{u_{i_1}(\mathbf{z})}{v_{i_1}(\mathbf{z})+\sum\limits_{i_2=1}^{2}\cfrac{u_{i_1,i_2}(\mathbf{z})}{v_{i_1,i_2}(\mathbf{z})+\ddots}} \tag{1.10}$$

converges to the finite value $f(\mathbf{z})$, *where* $v_0(\mathbf{z})$, $u_{i_1}(\mathbf{z})$, $v_{i_1}(\mathbf{z})$, $u_{i_1,i_2}(\mathbf{z})$, $v_{i_1,i_2}(\mathbf{z}),\ldots$ *are functions of* $\mathbf{z}$. *Let* $f_k(\mathbf{z})$ *denote the kth approximant of (1.10),* $k\ge 1$. *Then*

$$f(\mathbf{z})-f_k(\mathbf{z})$$

is called the truncation error of the kth approximant, and

$$|f(\mathbf{z})-f_k(\mathbf{z})|\le C_k(\mathbf{z})$$

is called a priori bound (or truncation error bound), where $C_k(\mathbf{z})\ge 0$ *and* $C_k(\mathbf{z})\to 0$ *as* $k\to+\infty$ *for all* $\mathbf{z}\in\mathfrak{D}$.

In this paper, we obtain truncation error bounds for branched continued fractions (1.6) and (1.8), and also establish domains of analytical continuation of the functions (1.3) and (1.4).

2. Main results

The following theorem holds.

Theorem 2.1. *Suppose that* α,γ *and* δ *are real numbers such that*

$$0<d_1\le\kappa,\ 0<c_k\le\kappa,\ k\ge 1, \tag{2.11}$$

where d_1, c_k, $k\ge 1$, *are defined by (1.9),* $\gamma\notin\{0,-1,-2,\ldots\}$, $\delta\notin\{0,-1,-2,\ldots\}$, κ *is a positive number. Then:*

(A) *The branched continued fraction (1.8) converges to a finite value* $f(\mathbf{z})$ *for each* $\mathbf{z}\in\mathfrak{D}_\eta$, *where*

$$\mathfrak{D}_\eta=\left\{\mathbf{z}\in\mathbb{R}^2:\ z_1\le 0,\ z_2\le\eta\right\},\ 0<\eta<\min\left\{\frac{1+\delta}{1-\alpha+\delta},\ 1\right\}. \tag{2.12}$$

(B) *The convergence is uniformly on every compact subset of* $\operatorname{Int}(\mathfrak{D}_\eta)$, *and* $f(\mathbf{z})$ *is analytic on* $\operatorname{Int}(\mathfrak{D}_\eta)$.

(C) *For each* $\mathbf{z}\in\mathfrak{D}_\eta$ *and for* $n\ge 3$

$$|f(\mathbf{z})-f_n(\mathbf{z})|\le\frac{|d_0||z_2|(|z_2|(1-z_2)+\kappa|z_1|)(1-z_2)^{-2}(1-z_2+\kappa|z_1|)^{-2}(\kappa|z_1|)^{n-1}}{(1-d_1z_2)((1-d_1z_2)(1-z_2)+\kappa|z_1|)((1-z_2)^2+\kappa|z_1|)^{n-3}},$$

where $f_n(\mathbf{z})$ *is the nth approximant of (1.8).*

(D) *The function* $f(\mathbf{z})$ *is an analytic continuation of (1.4) in* $\operatorname{Int}(\mathfrak{D}_\eta)$.

Note that the conditions (2.11) are satisfied if $0<\alpha<\delta+1$ and $\alpha<2\gamma-1$.

Proof of the Theorem 2.1. Let us use the idea of the proving Theorem 1 from [8]. First we set

$$U_n^{(n)}(\mathbf{z})=1,\ n\ge 1, \tag{2.13}$$

and

$$U_k^{(n)}(\mathbf{z}) = 1 - d_k z_2 - \cfrac{c_k z_1}{1 - d_{k+1} z_2 - \cfrac{c_{k+1} z_1}{1 - \ddots - d_{n-2} z_2 - \cfrac{c_{n-2} z_1}{1 - d_{n-1} z_2 - c_{n-1} z_1}}},$$

where $1 \le k \le n-1,\ n \ge 2$. Then

$$U_k^{(n)}(\mathbf{z}) = 1 - d_k z_2 - \frac{c_k z_1}{U_{k+1}^{(n)}(\mathbf{z})},\ 1 \le k \le n-1,\ n \ge 2, \tag{2.14}$$

and

$$f_n(\mathbf{z}) = 1 + \frac{d_0 z_2}{U_1^{(n)}(\mathbf{z})}, \quad n \ge 1.$$

Now, we will proof (A). Let $\mathbf{z}$ be an arbitrary fixed point in (2.12). From (2.11) it follows that the coefficients d_1, c_k, $k \ge 1$, are positive real numbers. Using inequalities from (2.12) and the relations (2.13) and (2.14), for any $n \ge 2$ we obtain

$$\begin{aligned} U_1^{(n)}(\mathbf{z}) &= 1 - d_1 z_2 - \frac{c_1 z_1}{U_2^{(n)}(\mathbf{z})} \\ &\ge 1 - d_1 z_2 \\ &\ge 1 - d_1 \eta \\ &> 0, \end{aligned}$$

and for arbitraries $n \ge 3$ and $2 \le k \le n-1$, we get

$$\begin{aligned} U_k^{(n)}(\mathbf{z}) &= 1 - d_k z_2 - \frac{c_k z_1}{U_{k+1}^{(n)}(\mathbf{z})} \\ &\ge 1 - z_2 \\ &\ge 1 - \eta \\ &> 0. \end{aligned}$$

This allows us to use the well-known formula form [4, p. 28]. Therefore, for $n \ge 2$ and $k \ge 1$

$$f_{n+k}(\mathbf{z}) - f_n(\mathbf{z}) = d_0 z_1^{n-1} z_2 \left(z_2 + \frac{c_n z_1}{U_{n+1}^{(n+k)}(\mathbf{z})} \right) \prod_{r=1}^{n-1} \frac{c_r}{U_r^{(n+k)}(\mathbf{z}) U_r^{(n)}(\mathbf{z})}$$

or the same

$$\begin{aligned} f_{n+k}(\mathbf{z}) - f_n(\mathbf{z}) &= \frac{d_0 z_1^{n-1} z_2}{U_1^{(q)}(\mathbf{z}) U_n^{(n+k)}(\mathbf{z})} \left(z_2 + \frac{c_n z_1}{U_{n+1}^{(n+k)}(\mathbf{z})} \right) \\ &\times \prod_{r=1}^{[(n-1)/2]} \frac{c_{2r-1}}{U_{2r-1}^{(p)}(\mathbf{z}) U_{2r}^{(p)}(\mathbf{z})} \prod_{r=1}^{[(n-2)/2]} \frac{c_{2r}}{U_{2r}^{(q)}(\mathbf{z}) U_{2r+1}^{(q)}(\mathbf{z})}, \end{aligned}$$

where $[.]$ denote integer part, $q = n+k$, $p = n$, if $n = 2s$, and $q = n$, $p = n+k$, if $n = 2s-1$, $s \ge 1$.

Now, for arbitraries $m \ge 2$ and $k \ge 2$ we have

$$\frac{d_0 z_2}{U_1^{(m)}(\mathbf{z})} \le \frac{|d_0||z_2|}{1 - d_1 z_2}, \frac{1}{U_m^{(m+k)}(\mathbf{z})} \left(z_2 + \frac{c_m z_1}{U_{m+1}^{(m+k)}(\mathbf{z})} \right) \le \frac{1}{1 - z_2} \left(|z_2| + \frac{\kappa |z_1|}{1 - z_2} \right),$$

for any $m \geq 2$ we obtain

$$\frac{c_1 z_1}{U_1^{(m+1)}(\mathbf{z})U_2^{(m+1)}(\mathbf{z})} = \frac{\dfrac{c_1 z_1}{U_2^{(m+1)}(\mathbf{z})}}{1 - d_1 z_2 - \dfrac{c_1 z_1}{U_2^{(m+1)}(\mathbf{z})}} \leq \frac{\dfrac{c_1 |z_1|}{U_2^{(m+1)}(\mathbf{z})}}{1 - d_1 z_2 + \dfrac{c_1 |z_1|}{U_2^{(m+1)}(\mathbf{z})}} \leq \frac{\kappa|z_1|}{(1 - d_1 z_2)(1 - z_2) + \kappa|z_1|},$$

for arbitraries $m \geq 3$ and $2 \leq k \leq m-1$ we get

$$\frac{c_k z_1}{U_k^{(m+1)}(\mathbf{z})U_{k+1}^{(m+1)}(\mathbf{z})} = \frac{\dfrac{c_k z_1}{U_{k+1}^{(m+1)}(\mathbf{z})}}{1 - z_2 - \dfrac{c_k z_1}{U_{k+1}^{(m+1)}(\mathbf{z})}} \leq \frac{\dfrac{c_k |z_1|}{U_{k+1}^{(m+1)}(\mathbf{z})}}{1 - z_2 + \dfrac{c_k |z_1|}{U_{k+1}^{(m+1)}(\mathbf{z})}} \leq \frac{\kappa|z_1|}{(1 - z_2)^2 + \kappa|z_1|},$$

and, finally, for any $m \geq 2$ we have

$$\frac{c_m z_1}{U_m^{(m+1)}(\mathbf{z})U_{m+1}^{(m+1)}(\mathbf{z})} = \frac{c_m z_1}{1 - z_2 - c_{m+1} z_1} \leq \frac{\kappa|z_1|}{1 - z_2 + \kappa|z_1|}.$$

Thus, for $n \geq 3$ and $k \geq 2$, we obtain

$$|f_{n+k}(\mathbf{z}) - f_n(\mathbf{z})| \leq \frac{|d_0||z_2|(|z_2|(1 - z_2) + \kappa|z_1|)(1 - z_2)^{-2}(1 - z_2 + \kappa|z_1|)^{-2}(\kappa|z_1|)^{n-1}}{(1 - d_1 z_2)((1 - d_1 z_2)(1 - z_2) + \kappa|z_1|)((1 - z_2)^2 + \kappa|z_1|)^{n-3}}. \tag{2.15}$$

It is obvious that for an arbitrary fixed $\mathbf{z} \in \mathfrak{D}_\eta$

$$\frac{|d_0||z_2|(|z_2|(1 - z_2) + \kappa|z_1|)(1 - z_2)^{-2}(1 - z_2 + \kappa|z_1|)^{-2}(\kappa|z_1|)^{n-1}}{(1 - d_1 z_2)((1 - d_1 z_2)(1 - z_2) + \kappa|z_1|)((1 - z_2)^2 + \kappa|z_1|)^{n-3}} \to 0$$

as $n \to +\infty$. Therefore, the arbitrariness of k it follows (A).

Next, we will proof (B). Let $\mathfrak{L}$ denote an arbitrary compact subset of $\mathrm{Int}(\mathfrak{D}_\eta)$. Then there exists an open ball of radius L such that for $n \geq 3$, $k \geq 2$ and for all $\mathbf{z} \in \mathfrak{L}$, we get

$$|f_{n+k}(\mathbf{z}) - f_n(\mathbf{z})| < \frac{|d_0|L(L(1 - \eta) + \kappa L)(1 - \eta)^{-2}(1 - \eta + \kappa L)^{-2}(\kappa L)^{n-1}}{(1 - d_1\eta)((1 - d_1\eta)(1 - \eta) + \kappa L)((1 - \eta)^2 + \kappa L)^{n-3}} = \frac{|d_0|(1 - \eta + \kappa)(1 - \eta)^{-2}(1 - \eta + \kappa L)^{-2}\kappa^{n-1}L^{n+1}}{(1 - d_1\eta)((1 - d_1\eta)(1 - \eta) + \kappa L)((1 - \eta)^2 + \kappa L)^{n-3}}.$$

Next, for arbitrary integer numbers q, p such that $q \geq 2$, $p \geq n \geq 2$, and for all $\mathbf{z} \in \mathfrak{L}$, we have

$$|f_{p+q}(\mathbf{z}) - f_p(\mathbf{z})| \leq |f_{p+q}(\mathbf{z}) - f_n(\mathbf{z})| + |f_p(\mathbf{z}) - f_n(\mathbf{z})|.$$

Furthermore, since

$$\frac{|d_0|(1-\eta+\kappa)(1-\eta)^{-2}(1-\eta+\kappa L)^{-2}\kappa^{n-1}L^{n+1}}{(1-d_1\eta)((1-d_1\eta)(1-\eta)+\kappa L)((1-\eta)^2+\kappa L)^{n-3}} \to 0$$

as $n \to +\infty$, it follows (B).

(C) follows directly from (2.15).

Finally, we will proof (D). It is obvious that

$$\frac{H_4(\alpha, \delta+1; \gamma, \delta; \mathbf{0})}{H_4(\alpha, \delta+2; \gamma, \delta+1; \mathbf{0})} = 1.$$

Then, there exists $0 < \varepsilon < 1$ such that (1.2) is an analytic function in the domain

$$\mathfrak{D}_{p,q,\varepsilon} = \left\{ \mathbf{z} \in \mathbb{R}^2 : \ -p\varepsilon < z_1 < 0, \ -q\varepsilon < z_2 < 0 \right\},$$

and

$$\mathfrak{D}_{p,q,\varepsilon} \subset (\mathfrak{D}_{p,q} \cap \mathrm{Int}(\mathfrak{D}_\eta)),$$

in particular,

$$\mathfrak{D}_{p,q,1/2} \subset (\mathfrak{D}_{p,q} \cap \mathrm{Int}(\mathfrak{D}_\eta)),$$

where $\mathfrak{D}_{p,q}$ is defined by (1.1).

Let $\mathbf{z}$ be an arbitrary fixed point in $\mathfrak{D}_{p,q,\varepsilon}$. It is clear that all elements of expansion (1.8) are positive numbers. This means that the approximants of (1.2) have the property of fork (see, [4, p. 29])

$$f_{2n}(\mathbf{z}) < f_{2n+2}(\mathbf{z}) < f_{2n+1}(\mathbf{z}) < f_{2n-1}(\mathbf{z}), \ n \geq 1$$

and, therefore, the sequences $\{f_{2n}(\mathbf{z})\}$ and $\{f_{2n-1}(\mathbf{z})\}$ converge to a finite value $f(\mathbf{z})$.

Let n be an arbitrary natural number. Consider the following expression

$$\frac{H_4(\alpha, \delta+1; \gamma, \delta; \mathbf{z})}{H_4(\alpha, \delta+2; \gamma, \delta+1; \mathbf{z})} - f_n(\mathbf{z}), \ n \geq 1,$$

where (see [1])

$$\frac{H_4(\alpha, \delta+1; \gamma, \delta; \mathbf{z})}{H_4(\alpha, \delta+2; \gamma, \delta+1; \mathbf{z})} = 1 + \cfrac{d_0 z_2}{1 - d_1 z_2 - \cfrac{c_1 z_1}{1 - \ddots - d_n z_2 - \cfrac{c_n z_1}{V_{n+1}^{(n+1)}(\mathbf{z})}}},$$

and

$$V_{n+1}^{(n+1)}(\mathbf{z}) = \frac{H_4(\alpha, \delta+n+2; \gamma, \delta+n+1; \mathbf{z})}{H_4(\alpha, \delta+n+3; \gamma, \delta+n+2; \mathbf{z})}.$$

Similar to (2.14), we have

$$V_k^{(n+1)}(\mathbf{z}) = 1 - d_k z_2 - \frac{c_k z_1}{V_{k+1}^{(n+1)}(\mathbf{z})}, \ 1 \leq k \leq n, \tag{2.16}$$

where

$$V_k^{(n+1)}(\mathbf{z}) = 1 - d_k z_2 - \cfrac{c_k z_1}{1 - d_{k+1} z_2 - \cfrac{c_{k+1} z_1}{1 - \ddots - d_n z_2 - \cfrac{c_n z_1}{V_{n+1}^{(n+1)}(\mathbf{z})}}}, \ 1 \leq k \leq n.$$

It is obvious that $U_k^{(n)}(\mathbf{z}) \neq 0$ and $V_k^{(n)}(\mathbf{z}) \neq 0$ for $1 \leq k \leq n$ and for $\mathbf{z} \in \mathfrak{D}_{p,q,\varepsilon}$. Using (2.13), (2.14), (2.16), and [4, Formula (3.3)], we get

$$\frac{H_4(\alpha, \delta+1; \gamma, \delta; \mathbf{z})}{H_4(\alpha, \delta+2; \gamma, \delta+1; \mathbf{z})} - f_n(\mathbf{z}) = d_0 z_1^{n-1} z_2 \left(z_2 + \frac{c_{n+1} z_1}{V_{n+1}^{(n+1)}(\mathbf{z})} \right) \prod_{r=1}^{n-1} \frac{c_r}{V_r^{(n+1)}(\mathbf{z}) U_r^{(n)}(\mathbf{z})}.$$

Then, for all $\mathbf{z} \in \mathfrak{D}_{p,q,\varepsilon}$ we have

$$f_{2n}(\mathbf{z}) < \frac{H_4(\alpha, \delta+1; \gamma, \delta; \mathbf{z})}{H_4(\alpha, \delta+2; \gamma, \delta+1; \mathbf{z})} < f_{2n-1}(\mathbf{z}).$$

Next, from the fork property of approximants of (1.2) it follows that for all $\mathbf{z} \in \mathfrak{D}_{p,q,\varepsilon}$

$$\lim_{n \to +\infty} f_{2n}(\mathbf{z}) = \lim_{n \to +\infty} f_{2n-1}(\mathbf{z}) = f(\mathbf{z})$$

and, therefore, for all $\mathbf{z} \in \mathfrak{D}_{p,q,\varepsilon}$,

$$f(\mathbf{z}) = \frac{H_4(\alpha, \delta+1; \gamma, \delta; \mathbf{z})}{H_4(\alpha, \delta+2; \gamma, \delta+1; \mathbf{z})}.$$

Finally, from [13, Theorem 3], it follows (D). □

The following result can be proved in much the same way as Theorem 2.1.

Theorem 2.2. *Let α, δ and γ be real numbers such that satisfy the inequalities*

$$\frac{\delta-\alpha}{\delta} > 0, \ 0 < b_k \leq \tau, \ k \geq 1,$$

where b_k, $k \geq 1$, are defined by (1.7), $\gamma \notin \{0, -1, -2, \ldots\}$, $\delta \notin \{0, -1, -2, \ldots\}$, τ is a positive number. Then:

(A) *The branched continued fraction (1.6) converges to a finite value $g(\mathbf{z})$ for each $\mathbf{z} \in \mathfrak{Q}_\eta$, where*

$$\mathfrak{Q}_\eta = \left\{ \mathbf{z} \in \mathbb{R}^2 : \ z_1 \leq 0, \ z_2 \leq \eta \right\}, \ 0 < \eta < 1. \tag{2.17}$$

(B) *The convergence is uniformly on every compact subset of the domain $\operatorname{Int}(\mathfrak{Q}_\eta)$, and $g(\mathbf{z})$ is analytic function on $\operatorname{Int}(\mathfrak{Q}_\eta)$.*

(C) *For each $\mathbf{z} \in \mathfrak{Q}_\eta$ and $n \geq 2$*

$$|g(\mathbf{z}) - g_n(\mathbf{z})| \leq \frac{(|z_2|(1-z_2) + \tau|z_1|)(\tau|z_1|)^n}{(1-z_2)^3(1-z_2+\tau|z_1|)((1-z_2)^2+\tau|z_1|)^{n-2}},$$

where $g_n(\mathbf{z})$ is the nth approximant of (1.6).

(D) *The function $g(\mathbf{z})$ is an analytic continuation of the function (1.3) in the domain $\operatorname{Int}(\mathfrak{Q}_\eta)$.*

By setting $\alpha = 0$ and replacing δ with $\delta - 1$ in Theorem 2.2, we have the following result.

Corollary 2.1. *Let δ and γ be a real number that satisfy condition*

$$0 < \frac{2}{\gamma} \leq \lambda, \ 0 < \frac{k(2\gamma-k-3)}{(\gamma+k-2)(\gamma+k-1)} \leq \lambda, \ k \geq 2,$$

herewith $\delta \notin \{1, 0, -1, -2, \ldots\}$, $\gamma \notin \{0, -1, -2, \ldots\}$, λ is a positive number. Then:

(A) *The branched continued fraction*

$$\cfrac{1}{1 - z_2 - \cfrac{b_1 z_1}{1 - z_2 - \cfrac{b_2 z_1}{1 - \ddots}}} \tag{2.18}$$

converges to a finite value $h(\mathbf{z})$ for each $\mathbf{z} \in \mathfrak{Q}_\eta$, where $\mathfrak{Q}_\eta$ is defined by (2.17).

(B) *The convergence is uniformly on every compact subset of the domain* $\mathrm{Int}(\mathfrak{Q}_\eta)$, *and* $h(\mathbf{z})$ *is analytic function on* $\mathrm{Int}(\mathfrak{Q}_\eta)$.

(C) *For each* $\mathbf{z} \in \mathfrak{Q}_\eta$ *and* $n \geq 3$

$$|h(\mathbf{z}) - h_n(\mathbf{z})| \leq \frac{(|z_2|(1-z_2)+\lambda|z_1|)(\lambda|z_1|)^{n-1}}{(1-z_2)^5(1-z_2+\lambda|z_1|)((1-z_2)^2+\lambda|z_1|)^{n-3}},$$

where $h_n(\mathbf{z})$ *is the* n*th approximant of (2.18).*

(D) *The function* $h(\mathbf{z})$ *is an analytic continuation of the function* $H_4(1,\delta;\gamma,\delta;\mathbf{z})$ *in the domain* $\mathrm{Int}(\mathfrak{Q}_\eta)$.

Note that in Corollary 2.1 and [8, Corollary 1] there are different conditions on the parameters δ and γ.

The truncation error bounds for branched continued fractions (1.5), (1.6) and (1.8) in $\mathbb{C}^2$ will be discussed in our next paper. The convergence of the expansions for functions (see [1])

$$\frac{H_4(\alpha,\beta;\gamma,\delta;\mathbf{z})}{H_4(\alpha+1,\beta;\gamma+1,\delta;\mathbf{z})},\ \frac{H_4(\alpha,\beta;\gamma,\delta;\mathbf{z})}{H_4(\alpha+1,\beta;\gamma,\delta+1;\mathbf{z})},\ \frac{H_4(\alpha,\beta;\gamma,\delta;\mathbf{z})}{H_4(\alpha,\beta+1;\gamma,\delta+1;\mathbf{z})}$$

remains open.

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Research Article

Maximum modulus of slice entire regular functions of quaternionic variable with bounded index

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ABSTRACT. The manuscript contains new results describing local behavior of slice entire regular functions of quaternionic variable. There is selected such their subclass as functions having bounded index and for these functions we describe uniform estimate of their maximum modulus in a disc of larger radius by their maximum modulus in a disc of lesser radius multiplied on some constant.

Keywords: Slice entire function, bounded index, maximum modulus, local behavior.

2020 Mathematics Subject Classification: 30G35.

1. INTRODUCTION

The paper is a continuation of investigations initiated in [1, 2]. Among them, paper [2] was an attempt to introduce a notion of bounded index for Fueter regular functions of quaternionic variable (see more details on the notion of regularity in [14, 19, 20]). The approach considers all possible partial derivatives in real components of the quaternionic variable and it is close to bounded index in joint variables [3, 6, 9]. It can also be applicable to other non-commutative algebras, for example, such as in [19, 18]. In the second paper [1], there was studied a similar problem for another concept of regularity – so-called slice regularity [15, 16, 17]. This notion is very flexible in the quaternionic analysis because it allows to deduce more quaternionic analogs of known statements from the complex analysis [13].

One should observe that the notion of slice holomorphy exists even in the multidimensional complex analysis [4, 8, 12]. However, in this context, a slice holomorphic function of several complex variables is understood to be a function that is holomorphic on all slices in one fixed direction, and in order to cover the whole n-dimensional complex space [7, 21] (either a unit ball [11], or unit polydisc [10]), we change the initial point of this slice in the form $z^0 + t\mathbf{b}$, where $z^0 \in \mathbb{C}^n$ is the start point, $\mathbf{b} \in \mathbb{C}^n$ is the fixed direction, and t is the variable parameter. In contrast, the quaternionic analysis considers slices of the form $x + Iy$, with $x, y \in \mathbb{R}$ and I is the variable point from the unit sphere of purely imaginary quaternions. In other words, the direction changes here, but for every fixed I it is possible to use classical results from the complex analysis.

Our goal is to combine the notion of slice regularity with the notion of bounded index and to deduce new results. It is important in view of known applications of functions having bounded index to analytic theory of differential equations [11, 12, 23].

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2. Main definitions and notations

We will use standard notations of quaternionic analysis from [1, 13]. Let $\mathbb{H}$ be the skew field of quaternions which is defined as $\mathbb{H} = \{g = x_0 + ix_1 + jx_2 + kx_3 \colon x_0, x_1, x_2, x_3 \in \mathbb{R}\}$, where the imaginary units i, j, k satisfy $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. It is a non-commutative field. We define the Euclidean norm on $\mathbb{H} : |q| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$. The quantities $Re\, q = x_0$, and $Im\, q = ix_1 + jx_2 + kx_3$, we call the real and imaginary parts of the quaternion $q = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$, respectively. The symbol $\mathbb{S}$ denotes the unit sphere (it is a sphere in $\mathbb{R}^3$ and it is a cylinder in $\mathbb{H}$, i.e., in $\mathbb{R}^4$) of purely imaginary quaternions, i.e., $\mathbb{S} = \{q = ix_1 + jx_2 + kx_3 \colon x_1^2 + x_2^2 + x_3^2 = 1\}$. One should observe that if $I \in \mathbb{S}$, then $I^2 = -1$. Given this, the elements of $\mathbb{S}$ are also called imaginary units. For any fixed $I \in \mathbb{S}$ we define the 'complex plane' $\mathbb{C}_I := \{x + Iy : x, y \in \mathbb{R}\}$. It is easy to check that $\mathbb{C}_I$ is isomorphic with the complex plane $\mathbb{C}$. Moreover, $\mathbb{H} = \bigcup_{I\in\mathbb{S}} \mathbb{C}_I$. In this case, the real axis $\{q\colon\ Im\, q = 0\}$ belongs to $\mathbb{C}_I$ for every $I \in \mathbb{S}$ and thus a real quaternion can be associated with any imaginary unit I. Any non-real quaternion $q = x_0 + ix_1 + jx_2 + kx_3$ is uniquely associated to the element $I_q \in \mathbb{S}$ defined by

$$I_q := \frac{ix_1 + jx_2 + kx_3}{|ix_1 + jx_2 + kx_3|}.$$

It is obvious that q belongs to the complex plane $\mathbb{C}_{I_q}$.

Let $f : \mathbb{H} \to \mathbb{H}$ be real differentiable. The function f is said to be (left) entire slice regular, if for every $I \in \mathbb{S}$ its restriction f_I to the complex plane $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ passing through origin and containing I and 1 satisfies $\bar{\partial}_I f(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + Iy) = 0$ on $\mathbb{C}_I$. The class of (left) slice regular functions on $\mathbb{H}$ will be denoted by $\mathcal{R}(\mathbb{H})$. Analogously, a function f is said to be right entire slice regular in $\mathbb{H}$ if $(f\bar{\partial}_I)(x + Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy)I\right) = 0$ on $\mathbb{C}_I$. Let $f \in \mathcal{R}(\mathbb{H})$. The so-called left I-derivative of f at a point $q = x + Iy$ is defined by $\partial_I f_I(x+Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) - I\frac{\partial}{\partial y} f_I(x + Iy)\right)$ and the right I-derivative of f at $q = x + Iy$ is defined by $\partial_I f_I(x+Iy) := \frac{1}{2}\left(\frac{\partial}{\partial x} f_I(x + Iy) - \frac{\partial}{\partial y} f_I(x + Iy)I\right)$. Let us now introduce another suitable notion of derivative. The slice derivative $\partial_s f$ of f, is defined by:

$$\partial_s(f)(q) = \begin{cases} \partial_I(f)(q), & \text{if } q = x + Iy,\ y \neq 0, \\ \frac{\partial f}{\partial x}(x), & \text{if } q = x \in \mathbb{R}. \end{cases}$$

We will often write $f'(q)$ instead of $\partial_s f(q)$. The k-th derivative of $f \in \mathcal{R}(\mathbb{H})$ is defined recursively as $f^{(k)}(q) = (f^{(k-1)}(q))'$. It is important to note that if $f(q)$ is a slice regular function then also $f'(q)$ is a slice regular function.

A function $f \in \mathcal{R}(\mathbb{H})$ is called a function of bounded index (see [1]), if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and every $q \in \mathbb{H}$ the following inequality is valid

$$\frac{|f^{(m)}(q)|}{m!} \leq \max\left\{\frac{|f^{(k)}(q)|}{k!} : 0 \leq k \leq m_0\right\}. \tag{2.1}$$

The least such integer m_0 is called the index of the entire slice regular function f and is denoted by $N(f) = m_0$. As an addendum to notion of bounded index there is known a notion of bounded $\mathfrak{M}$-index [22] which allows to examine analytic functions with unbounded multiplicities of zeros.

There were obtained two following criteria of index boundedness for slice entire regular functions in [1].

Theorem 2.1 ([1]). *A function $f \in \mathcal{R}(\mathbb{H})$ is of bounded index if and only if for every $\eta > 0$ there exist $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that for any $I \in \mathbb{S}$ and for every $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}$ (or, equivalently, for any $q = x_0 + Iy_0 \in \mathbb{H}$) there exists $k_0 = k_0(q) = k_0(x_0, y_0, I) \in \mathbb{Z}_+$ with $0 \leq k_0 \leq n_0$ and the following inequality holds*

$$\max\{|f_I^{(k_0)}(x + Iy)| : \sqrt{(x - x_0)^2 + (y - y_0)^2} \leq \eta\} \leq P_1 |f_I^{(k_0)}(x_0 + Iy_0)|. \tag{2.2}$$

Theorem 2.2 ([1]). *Let $f \in \mathcal{R}(\mathbb{H})$. If there exist $\eta > 0$, $n_0 = n_0(\eta) \in \mathbb{Z}_+$ and $P_1 = P_1(\eta) \geq 1$ such that for any $I \in \mathbb{S}$ and for every $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}$ there exists $k_0 = k_0(q) = k_0(x_0, y_0, I) \in \mathbb{Z}_+$, with $0 \leq k_0 \leq n_0$, and (2.2) is satisfied, then the function f has bounded index.*

3. Estimate of the maximum modulus on a larger circle by the maximum modulus on a smaller circle.

Now we will try to estimate the maximum modulus of entire slice regular function on a circle. Using Theorem 2.1, we prove a criterion of index boundedness in direction. This result was announced under a talk at the International Workshop on Modern Problems of Analysis, Optimization, Approximation and Their Applications [5].

Theorem 3.3. *A function $f \in \mathcal{R}(\mathbb{H})$ is of bounded index if and only if for each pair of radii r_1 and r_2 with $0 < r_1 < r_2 < \infty$ there exists number $P_1 \geq 1$ depending only on r_1 and r_2, i.e., $P_1 = P_1(r_1, r_2)$, such that for any $I \in \mathbb{S}$ and for every $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}$ (or, equivalently, for any $q = x_0 + Iy_0 \in \mathbb{H}$) the following inequality holds*

$$\begin{aligned} &\max\{|f_I(x + Iy)| : \sqrt{(x - x_0)^2 + (y - y_0)^2} = r_1\} \\ \leq &P_1 \max\{|f_I(x + Iy)| : \sqrt{(x - x_0)^2 + (y - y_0)^2} = r_2\}, \end{aligned} \tag{3.3}$$

where maxima in equation (3.3) are taken over circles within the corresponding complex slice.

Proof. **Necessity.** Let f be of bounded index and $N(f) \equiv N < +\infty$. We will prove the converse statement and introduce a proof by contradiction. Suppose that there exist radii r_1 and r_2, $0 < r_1 < r_2 < \infty$, such that for any constant $P \geq 1$ there exist an imaginary unit $I_* \in \mathbb{S}$ and appropriate $x_* \in \mathbb{R}$, $y_* \in \mathbb{R}$ (or, equivalently, some quaternionic point $q_* = x_* + I_* y_* \in \mathbb{H}$) providing validity of the following inequality

$$\begin{aligned} &\max\{|f_{I_*}(x + I_* y)| : \sqrt{(x - x_*)^2 + (y - y_*)^2} = r_2\} \\ > &P \max\{|f_{I_*}(x + I_* y)| : \sqrt{(x - x_*)^2 + (y - y_*)^2} = r_1\}. \end{aligned} \tag{3.4}$$

In Theorem 2.1, we put $\eta = r_2$ and apply this theorem. Then there exist $n_0 = n_0(r_2) \in \mathbb{Z}_+$ and $P_1 = P_1(r_2) \geq 1$ such that for every $I \in \mathbb{S}$ and for every $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}_+$ and some $k_0 = k_0(q) = k_0(x_0, y_0, I) \in \mathbb{Z}_+$, $0 \leq k_0 \leq n_0$, one has

$$\max\{|f_I^{(k_0)}(x + Iy)| : \sqrt{(x - x_0)^2 + (y - y_0)^2} \leq r_2\} \leq P_1 |f_I^{(k_0)}(x_0 + Iy_0)|. \tag{3.5}$$

In view of the arbitrariness of $I \in \mathbb{S}$, $x_0 \in \mathbb{R}$, $y_0 \in \mathbb{R}_+$, in (3.5) we substitute $I = I_*$, $x_0 = x_*$, $y_0 = y_*$ from (3.4). The choice of I does not affect the index $N(f)$. Then for some k_0 it yields

$$\max\{|f_{I_*}^{(k_0)}(x + I_* y)| : \sqrt{(x - x_*)^2 + (y - y_*)^2} \leq r_2\} \leq P_1 |f_{I_*}^{(k_0)}(x_* + I_* y_*)|. \tag{3.6}$$

For further proof, we need the Maximum Modulus Principle for slice entire regular functions. Let $U \subset \mathbb{H}$. By Definition 2.4 from [13] the domain U is a slice domain if it is a connected set, whose intersection with every complex plane $\mathbb{C}_I$ is connected. By the Maximum Modulus Principle for slice domains (see Theorem 3.12 in [13, P. 45]), if U is a slice domain, $f : U \to \mathbb{H}$

is slice regular and $|f|$ has a relative maximum at $p \in U$, then f is constant. The principle is applicable in our case because for given $I \in \mathbb{S}$ the domain

$$\left\{x+Iy\colon \sqrt{(x-x_0)^2+(y-y_0)^2} \le \frac{p\eta}{g(\eta)}\right\}$$

is a slice domain. The Maximum Modulus Principle holds for each slice $\mathbb{C}_I$, not for the entire quaternionic space $\mathbb{H}$.

First, we consider the case $k_0 = 0$. Then (3.6) and double application of the Maximum Modulus Principle yields

$$\begin{aligned}
&\max\left\{|f_{I_*}(x+I_*y)| : \sqrt{(x-x_*)^2+(y-y_*)^2} = r_2\right\}\\
=&\max\left\{|f_{I_*}(x+I_*y)| : \sqrt{(x-x_*)^2+(y-y_*)^2} \le r_2\right\}\\
\le& P_1|f_{I_*}(x_*+I_*y_*)|\\
\le& P_1\max\left\{|f_{I_*}(x+I_*y)| : \sqrt{(x-x_*)^2+(y-y_*)^2} \le r_1\right\}\\
=& P_1\max\left\{|f_{I_*}(x+I_*y)| : \sqrt{(x-x_*)^2+(y-y_*)^2} = r_1\right\}.
\end{aligned}$$

Hence, estimate (3.3) holds and the necessity is proved in the case.

Suppose that $k_0 > 0$. Put in (3.4)

$$P = n_0!\left(\frac{r_2}{r_1}\right)^{n_0}\left(P_1+\frac{r_1}{r_2-r_1}\right)+1. \tag{3.7}$$

Let us introduce auxiliary maximum modulus points on the slice discs. We assume $\mathring{q} = \mathring{x} + I_*\mathring{y} \in H$ is such that

$$|f_I(\mathring{x}+I_*\mathring{y})| = \max\left\{|f_{I_*}(x+I_*y)| : \sqrt{(x-x_*)^2+(y-y_*)^2} = r_1\right\},$$

and $q_{*j} = x_{*j} + I_*y_{*j} \in \mathbb{H}$ is such that

$$|f^{(j)}_{I_*}(x_{*j}+I_*y_{*j})| = \max\{|f^{(j)}_{I_*}(x+I_*y)| : \sqrt{(x-x_*)^2+(y-y_*)^2} = r_2\},$$

$j \in \mathbb{Z}_+$.

In the case $|f_{I_*}(\mathring{x}+I_*\mathring{y})| = 0$ it follows that for all $x \in \mathbb{R}$ and $y \in \mathbb{R}$ with $(x-x_*)^2+(y-y_*)^2 = r_1^2$ one has $f_{I_*}(\mathring{x}+I_*\mathring{y}) \equiv 0$. Further let us remind Identity Principle for slice entire regular functions (see [13, Theorem 2.3, p. 11]):

Let $f : U \to \mathbb{H}$ be a slice regular function on a slice domain U, $Z_f = \{q \in U : f(q) = 0\}$ be the zero set of f. If there exists $I \in \mathbb{S}$ such that $\mathbb{C}_I \cap Z_f$ has an accumulation point, then $f \equiv 0$ on U.

By the Identity Principle, we obtain $f(q) \equiv 0$ which contradicts (3.4).

So, let $|f_{I_*}(\mathring{x}+I_*\mathring{y})| > 0$. We will use the following Cauchy estimates for slice entire regular functions. Let us cite Proposition 3.1 in [13, P. 32]. Let $F : U \to \mathbb{H}$ be a slice regular function and let $q \in U \cap \mathbb{C}_I$. For all discs $B_I(q,R) = B(q,R) \cap \mathbb{C}_I$, $R > 0$ such that $B_I(q,R) \subset U \cap \mathbb{C}_I$ the following formula holds:

$$|F^{(n)}(q)| \le \frac{n!}{R^n}\max_{s\in\partial B_I(q,R)}|F(s)|. \tag{3.8}$$

By Cauchy's inequality for $n=j$, $R=r_1$ and $q=x_*+I_*y_*$ from (3.8) we obtain

$$\frac{|f^{(j)}_{I_*}(x_*+I_*y_*)|}{j!}\leq\left(\frac{1}{r_1}\right)^j\max\left\{|f_{I_*}(x+I_*y)|:\sqrt{(x-x_*)^2+(y-y_*)^2}=r_1\right\}$$

$$=\left(\frac{1}{r_1}\right)^j|f_{I_*}(\mathring{x}+I_*\mathring{y})|,\quad j\in\mathbb{Z}_+,\tag{3.9}$$

and

$$\left|f^{(j)}_{I_*}(x_{*j}+I_*y_{*j})-f^{(j)}_{I_*}(x_*+I_*y_*)\right|=\left|\int_{x_*+I_*y_*}^{x_{*j}+I_*y_{*j}}f^{(j+1)}_{I_*}(t)\,dt\right|$$

$$\leq\left|f_{I_*}(x_{*(j+1)}+I_*y_{*(j+1)})\right|r_2.\tag{3.10}$$

From (3.9) and (3.10) we have

$$|f^{(j+1)}_{I_*}(x_{*(j+1)}+I_*y_{*(j+1)})|\geq\frac{1}{r_2}\left\{f^{(j)}_{I_*}(x_{*j}+I_*y_{*j})|-|f^{(j)}_{I_*}(x_*+I_*y_*)|\right\}$$

$$\geq\frac{1}{r_2}\left|f^{(j)}_{I_*}(x_{*j}+I_*y_{*j})\right|-\frac{j!}{r_2(r_1)^j}|f_{I_*}(\mathring{x}+I_*\mathring{y})|,$$

where $j\in\mathbb{Z}_+$. Hence, for $k_0\geq1$ we get

$$|f^{(k_0)}_{I_*}(x_{*k_0}+I_*y_{*k_0})|\geq\frac{1}{r_2}|f^{(k_0-1)}_{I_*}(x_{*(k_0-1)}+I_*y_{*(k_0-1)})|-\frac{(k_0-1)!}{r_2(r_1)^{k_0-1}}|f_{I_*}(\mathring{x}+I_*\mathring{y})|$$

$$\geq\ldots\geq\frac{1}{(r_2)^{k_0}}|f_{I_*}(x_{*0}+I_*y_{*0})|$$

$$-\left(\frac{0!}{(r_2)^{k_0}}+\frac{1!}{(r_2)^{k_0-1}r_1}+\ldots+\frac{(k_0-1)!}{r_2(r_1)^{k_0-1}}\right)|f^j_{I_*}(\mathring{x}+I_*\mathring{y})|$$

$$=\frac{1}{(r_2)^{k_0}}|f_{I_*}(\mathring{x}+I_*\mathring{y})|\left(\frac{|f_{I_*}(x_{*0}+I_*y_{*0})|}{|f_{I_*}(\mathring{x}+I_*\mathring{y})|}-\sum_{j=0}^{k_0-1}j!\left(\frac{r_2}{r_1}\right)^j\right).\tag{3.11}$$

In view of (3.4), we have $|f_{I_*}(x_{*0}+I_*y_{*0})|/|f_{I_*}(\mathring{x}+I_*\mathring{y})|>P$. Besides, this inequality holds

$$\sum_{j=0}^{k_0-1}j!\left(\frac{r_2}{r_1}\right)^j\leq k_0!\left(\frac{(r_2/r_1)^{k_0}-1}{r_2/r_1-1}\right)\leq n_0!\frac{r_1}{r_2-r_1}\left(\frac{r_2}{r_1}\right)^{n_0}.$$

Applying (3.7), we obtain

$$\frac{|f_{I_*}(x_{*0}+I_*y_{*0})|}{|f_{I_*}(\mathring{x}+I_*\mathring{y})|}-\sum_{j=0}^{k_0-1}j!\frac{r_2^j}{r_1^j}>P-\frac{n_0!r_1}{r_2-r_1}\left(\frac{r_2}{r_1}\right)^{n_0}=n_0!\left(\frac{r_2}{r_1}\right)^{n_0}P_0+1>1.$$

It follows from (3.11), (3.6) and (3.9) that

$$\left|f^{(k_0)}_{I_*}(x_{*k_0}+I_*y_{*k_0})\right|>\frac{1}{(r_2)^{k_0}}\left(P-n_0!\frac{r_1}{r_2-r_1}\left(\frac{r_2}{r_1}\right)^{n_0}\right)(r_1)^{k_0}\frac{|f^{(k_0)}_{I_*}(x_*+I_*y_*)|}{k_0!}$$

$$\geq\left(\frac{r_1}{r_2}\right)^{n_0}\left(P-n_0!\frac{r_1}{r_2-r_1}\left(\frac{r_2}{r_1}\right)^{n_0}\right)\frac{|f^{(k_0)}_{I_*}(x_{*k_0}+I_*y_{*k_0})|}{n_0!P_1}.$$

Hence, $P<n_0!\left(\frac{r_2}{r_1}\right)^{n_0}\left(P_1+\frac{r_1}{r_2-r_1}\right)$, which contradicts (3.7).

Sufficiency. We choose any two numbers $r_1 \in (0,1)$ and $r_2 \in (1,+\infty)$. For given $q_0 = x_0 + I_0 y_0 \in \mathbb{H}$ we expand the function $f_I(x+Iy)$ in a power series by powers $x+Iy$

$$f_I(x+Iy) = \sum_{m=0}^{\infty} b_m(x_0+Iy_0)(x+Iy-(x_0+Iy_0))^m, \quad b_m(x_0+Iy_0) = \frac{f_I^{(m)}(x_0+Iy_0)}{m!}.$$

Since f is a slice entire regular function, the series uniformly converge in any disc $\{x+Iy : (x-x_0)^2+(y-y_0)^2 \le r^2\}$. For $r>0$, we denote

$$\begin{aligned} M(r,q_0,f) &= \max\{|f_I(x+Iy)| : (x-x_0)^2+(y-y_0)^2 = r^2\}, \\ \mu(r,q_0,f) &= \max\{|b_m(q_0)|r^m : m \ge 0\}, \\ \nu(r,q_0,f) &= \max\{m : |b_m(q_0)|r^m = \mu(r,q_0,f)\}. \end{aligned}$$

By Cauchy's inequality (3.8) one has

$$\begin{aligned} \mu(r,q_0,f) &= \max\{|b_m(q_0)|r^m : m \ge 0\} \\ &= \max\left\{\frac{|f_I^{(m)}(x_0+Iy_0)|}{m!} r^m : m \ge 0\right\} \le M(r,q_0,f). \end{aligned}$$

But for any $r>0$, we have

$$M(r_1 r, q_0, f) \le \sum_{m=0}^{\infty} |b_m(q_0)| r^m r_1^m \le \mu(r,q_0,f) \sum_{m=0}^{\infty} r_1^m = \frac{\mu(r,q_0,f)}{1-r_1}$$

and since $\nu(r,q^0,f)$ is monotone in r, we deduce

$$\ln \mu(r_2 r, q_0, f) - \ln \mu(r,q_0,f) = \int_r^{r_2 r} \frac{\nu(t,q_0,f)}{t} dt \ge \nu(r,q_0,f) \ln r_2.$$

Hence,

$$\begin{aligned} \nu(r,q_0,f) \le& \frac{1}{\ln r_2}(\ln \mu(r_2 r,q_0,f) - \ln \mu(r,q_0,f)) \\ \le& \frac{1}{\ln r_2}\{\ln M(r_2 r,q_0,f) - \ln((1-r_1)M(r_1 r,q_0,f))\} \\ =& -\frac{\ln(1-r_1)}{\ln r_2} + \frac{1}{\ln r_2}\{\ln M(r_2 r,q_0,f) - \ln M(r_1 r,q_0,f)\} \end{aligned} \tag{3.12}$$

Let $N(q_0,f)$ be the index of the function f at the point q_0, i.e., $N(q^0,f)$ is the smallest number m_0 for which inequality (2.1) holds with $q=q_0$. It is obvious that

$$N(q_0,f) \le \nu(1,q_0,f).$$

However, inequality (2.2) can be written in the following form

$$M(r_2,q_0,f) \le P_1(r_1,r_2) M(r_1,q_0,f).$$

Thus, from (3.12) for $r=1$ we obtain

$$N(q^0,f) \le -\frac{\ln(1-r_1)}{\ln r_2} + \frac{\ln P_1(r_1,r_2)}{\ln r_2}$$

for every $q_0 \in \mathbb{H}$, i.e.,

$$N(f) \le -\frac{\ln(1-r_1)}{\ln r_2} + \frac{\ln P_1(r_1,r_2)}{\ln r_2}.$$

The obtained bound does not depend on the point q_0. Thus, it justifies the definition of the global index. This completes the proof of Theorem 3.3. ☐

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Research Article

About Borel type relation for some positive integrals

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ABSTRACT. The manuscript contains new results describing asymptotic behavior of functions which are represented by integrals of the form $F(x) = \int_0^{+\infty} a(t)f(x+t)\nu(dt)$, where ν is locally finite measure on $\mathbb{R}_+$, a is positive ν-measurable function, f is positive and increasing to $+\infty$ in $[0,+\infty)$ function such that $f(0) = 1$ and $\ln f(x)$ is convex on the interval $[0,+\infty)$ function. The obtained main result was applied to the study of the stability of the maximum term of the series of the form $F(x) = \sum\limits_{n=0}^{\infty} a_n f(\lambda_n + x)$, $\quad a_n \geq 0\ (n \geq 0)$.

Keywords: Maximal term, functional series, Borel relation, stability of maximal term.

2020 Mathematics Subject Classification: 30G35.

1. INTRODUCTION

Everywhere in the text below we suppose that $\lambda = (\lambda_n)$ is a positive non-decreasing sequence, i.e., $\lambda_0 = 0$ and $\lambda_n \to +\infty$ as $n \to +\infty$, and $f\colon \mathbb{R}_+ := (0,+\infty) \to \mathbb{R}_+$ is a positive Borel function. In addition, let ν be a Borel measure on the σ-algebra $\mathcal{B}(\mathbb{R}_+)$ of Borel sets from $\mathbb{R}_+$. We also denote by $I(\nu, f)$ the class of functions $F\colon \mathbb{R}_+ \to \mathbb{R}_+$ represented for each $x \in \mathbb{R}$ by the integral of the form

$$F(x) = \int_0^{+\infty} a(t)f(tx)\nu(dt), \tag{1.1}$$

where $a\colon [0,+\infty) \to [0,+\infty)$ is a Borel function. In the case $\nu(dt) = dn_\lambda(t)$, where $n(t) = \sum_{\lambda_n \leq t} 1$ is the counting function of the sequence λ, we obtain the class $D(\lambda, f) := I(\nu, f)$ of series of the form

$$F(x) = \sum_{n=0}^{\infty} a_n f(\lambda_n x), \quad a_n \geq 0\ (n \geq 0), \tag{1.2}$$

where $a_n = a(\lambda_n) \geq 0\ (n \geq 0)$. For the functions $a(t)\colon [0,+\infty) \to [0,+\infty)$, $f(t)\colon [0,+\infty) \to [0,+\infty)$, a measure ν and every $x > 0$, we denote

$$\mu_*(x) = \sup\{a(t)f(tx)\colon t \in \text{supp}\ \nu\}.$$

Remark 1.1. supp $dn_\lambda = \{\lambda_n\colon n \geq 0\}$. *Therefore,*

$$\mu_*(x) = \sup\{a_n f(t\lambda_n)\colon n \geq 0\} := \mu(x, F)$$

is the maximal term of the series (1.2).

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In papers [7, 2], the conditions were obtained under which, the Borel-type asymptotic relation

$$\ln F(x) = (1 + o(1)) \ln \mu_*(x) \tag{1.3}$$

holds as $x \to +\infty$ for series of the form (1.2) outside some set of finite Lebesgue measure, where f is a positive functions on $\mathbb{R}_+$, such that the auxiliary function $y = \ln f(x) \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a convex function on $\mathbb{R}_+$.

In particular, in paper [2] the following theorem was proved.

Theorem 1.1 ([2]). *If condition*

$$\int_0^{+\infty} t^{-2} \ln \nu_0(t)\, dt < +\infty \tag{1.4}$$

holds with $\nu_0(t) = \nu\{u \geq 0 : \ln f(u) \leq t\}$, then for every function $F \in I(\nu, f)$ there exists a set E of finite Lebesgue measure such that the asymptotic relation (1.3) holds as $x \to +\infty$ $(x \notin E)$.

In the case $\nu(dt) = dn_\lambda(t)$, condition (1.4) is equivalent (see results for positive integrals [6] and for the Dirichlet series [2, 4, 5]) to the condition

$$\sum_{n=1}^{+\infty} \frac{1}{n \ln f(\lambda_n)} < +\infty. \tag{1.5}$$

Therefore, from Theorem 1.1 we obtain:

Theorem 1.2 ([7]). *Let $f \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function such that the function $\ln f(x) \colon \mathbb{R}_+ \to \mathbb{R}_+$ is a convex function and a non-negative sequence (λ_n) be such that $0 \leq \lambda_n \uparrow +\infty$ $(0 \leq n \uparrow +\infty)$. If a function F represented on $(0, +\infty)$ by a series of form (1.2) and condition (1.5) holds, then there exists a set $E \subset (0, +\infty)$ of finite Lebesgue measure such that asymptotic relation (1.3) holds as $x \to +\infty$ $(x \notin E)$.*

In [6], a similar result about the Borel-type relation (1.3) was obtained for more general positive integrals of the form

$$F(x) = \int_0^{+\infty} a(t) f(tx + \beta(t)\tau(x)) \nu(dt), \tag{1.6}$$

which are, in particular, generalizations of the Taylor-Dirichlet type series. Here $a(t)$, $\beta(t)$ are the positive Borel functions and a function f is the same as above.

Theorem 1.3 ([6]). *Let F be a function of form (1.6). If function $\tau(x) \colon \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable on $[x_0, +\infty)$ such that $\tau'(x) \geq 1$ $(x \geq x_0)$ and the condition (1.4) holds with*

$$\nu_0(t) = \nu\big(\{u \geq 0 \colon\ \ln f(u + \beta(u)) \leq t\}\big),$$

then for every function F of form (1.6) there exists a set E of finite Lebesgue measure such that the asymptotic relation (1.3) holds as $x \to +\infty$ $(x \notin E)$.

In article [8] (see also [14, 15, 16]), there was considered regularly convergent series of the form

$$G(z) = \sum_{n=0}^{+\infty} b_n g(z\beta_n), \tag{1.7}$$

that is $\mathfrak{M}_G(r) := \sum_{n=0}^{+\infty} |b_n| M_g(r\beta_n) < +\infty$ for all $r > 0$, where $g(z)$ is some entire function, (β_n) is a given non-negative sequence such that $\beta_n \uparrow +\infty$ $(n \uparrow +\infty)$ and $M_g(r) = \max\{|g(z)| \colon |z| = r\}$. If we now denote $F(x) = \mathfrak{M}_G(x)$, $f(x) = M_g(x)$, $\lambda_n = \beta_n$, then we obtain a series of form

(1.2), where the function $\ln f(x)$ is logarithmically convex, that is, the function $h(x) := \ln f(e^x)$ is a convex function on $\mathbb{R}$. The growth and expansion properties of similar series were also studied in [9, 10, 17] the Hadamard composition of such series was considered in [11], the boundedness of the $\ell - \mathfrak{M}$-index for them recently was examined in [12, 13].

For the function $f(x) = e^x$ we obtain an entire Dirichlet series $F(x) = \sum_{n=1}^{\infty} a_n e^{x\lambda_n}$. Then, provided $\sum_{n=1}^{+\infty} 1/\big(n \ln f(\lambda_n)\big) = \sum_{n=1}^{+\infty} 1/\big(n\lambda_n\big) < +\infty$, from Theorem 1.2 it follows the statement of theorem in [4] for entire Dirichlet series. Note that in the case where the function $\ln f(x)$ is not convex, we cannot apply the statement of Theorem 1.2. The following conjecture from the article [8] is particularly relevant to this circumstance.

Let us denote $\Gamma_f(x) = x(\ln f(x))'_+$.

Conjecture 1.1 ([8]). *If*

$$\sum_{n=1}^{\infty} \frac{1}{n\Gamma_f(\lambda_n)} < +\infty, \tag{1.8}$$

then asymptotic relation (1.3) *holds as $x \to +\infty$ outside some exceptional set E such that $\int_E \Gamma_f(x)\frac{dx}{x} < +\infty$ for every functions F of form* (1.2).

For each entire transcendental function f we get $\Gamma_f(r) = r(\ln M_f(r))'_+ \nearrow +\infty$ $(r \to +\infty)$. Therefore, from condition $\int_E \Gamma_f(x)\frac{dx}{x} < +\infty$ it follows that $\int_E \frac{dx}{x} < +\infty$, that is, a set E has finite logarithmic measure (it was incorrectly written in [3] that the set E has finite Lebesgue measure).

Note that in different cases the condition (1.5) can be either weaker than the condition (1.8) or stronger. Indeed, if we choose ([3]) $\ln f(t) = (\ln t)^{1+\varrho}$, $\varrho > 0$, then $\Gamma_f(r) = (1+\varrho)(\ln t)^{\varrho}$ and the condition (1.5) is weaker than the condition (1.8). However, in the case of $\Gamma_f(r)/r \nearrow +\infty$ $(r_0 \le r \to +\infty)$, we have

$$\ln f(x) - \ln f(r_0) = \int_{r_0}^{r} \frac{\Gamma_f(t)}{t} dt \le \Gamma_f(r) \quad (r \to +\infty).$$

Therefore, the condition (1.5) is, in general, stronger than the condition (1.8). However, there is a nuance. Under the conditions of Sheremeta's conjecture the function $\ln f(x)$ should be considered logarithmically convex. And in the statement of Theorem 1.2, the function $\ln f(x)$ is convex. One should observe that the condition $\Gamma_f(r)/r \nearrow +\infty$ $(r_0 \le r \to +\infty)$ means that as in Theorem 1.2 the function $\ln f(x)$ is convex. Let us now formulate the following conjecture.

Conjecture 1.2. *The statement of Conjecture 1.1* ([8]) *holds in case if a function f is such that the function $\ln f(x)$ is convex.*

Similar conjecture we formulate also about the Borel relation for the class $I(\nu, f)$.

Conjecture 1.3. *If condition* (1.4) *holds with $\nu_0(t) = \nu\{u \ge 0 : \ln f(u) \le t\}$ and a function f is such that the function $\ln f(x)$ is convex, then for every function $F \in I(\nu, f)$ there exists a set E such that $\int_{x_0}^{+\infty} \frac{\Gamma_f(x)}{x} dx < +\infty$ and the asymptotic relation* (1.3) *holds as $x \to +\infty$ $(x \notin E)$.*

2. Main Result

Let $I_+(\nu, f)$ be the class of functions $F: \mathbb{R}_+ \to \mathbb{R}_+$ represented by integrals of the form

$$F(x) = \int_0^{+\infty} a(t) f(x+t) \nu(dt),$$

where ν is locally finite measure on $\mathbb{R}_+$, a is positive ν-measurable function, f is positive and increasing to $+\infty$ in $[0,+\infty)$ function such that $f(0)=1$ and $\ln f(x)$ is convex on the interval $[0,+\infty)$ function.

The following statement is an analogue of Theorem 3.1 for integrals from the class $I_+(\nu,f)$ and gives a positive answer to Conjecture 1 from [1].

Proposition 2.1. *If condition*

$$\int_0^{+\infty} t^{-2}\ln\nu_0(t)\,dt<+\infty \tag{2.9}$$

holds with $\nu_0(t)=\nu\{u\geq 0:(\ln f(u))'\leq t\}$, then for every function $F\in I_+(\nu,f)$ there exists a set E of finite Lebesgue measure such that the asymptotic relation

$$\ln F(x)\leq (1+o(1))\ln\mu(x,F)$$

holds as $x\to+\infty$, $(x\notin E)$, where $\mu(x,F)=\sup\{a(t)f(x+t): x\in \textit{supp}\,\nu\}$ and supp ν is the support of the measure ν.

Proof. Denote $g(x)=\ln F(x)$. In the following proof, we reason similarly to the proof of Corollary 3.1 in [3]. We assume that $f'(x)$ denotes the right-hand derivative.

Since $g_0(x):=\ln f(x)$ is convex function on $\mathbb{R}_+$, $(\ln f(t))'\leq(\ln f(u))'\big|_{u=x+t}$ and $f'(x+t)>0$ for fixed $x>0$ and for every $t>0$. Then for fixed $x>0$

$$G:=G_x=\left\{t>0\colon (\ln f(u))'\Big|_{u=x+t}\leq 2g'(x)\right\}\subset\{t>0\colon(\ln f(t))'\leq 2g'(x)\}:=G_0,$$

where $g(x):=\ln F(x)$. Hence,

$$\nu(G)\leq\nu(G_0)=\nu_0(2g'(x)).$$

For $x>0$ and $t\notin G_x$, we have $(\ln f(u))'\Big|_{u=x+t}>2g'(x)$, i.e.,

$$\begin{aligned}\int\limits_{\mathbb{R}_+\setminus G} a(t)f(t+x)\nu(dt)&=\int\limits_{\mathbb{R}_+\setminus G} a(t)\frac{f'(t+x)}{(\ln f(u))'\big|_{u=t+x}}\nu(dt)\\&\leq\frac{1}{2g'(x)}\int\limits_{\mathbb{R}_+\setminus G} a(t)f'(t+x)\nu(dt)\\&\leq\frac{F(x)}{2F'(x)}\int\limits_{\mathbb{R}_+} a(t)f'(t+x)\nu(dt)=\frac{F(x)}{2},\end{aligned}$$

because $F'(x)=\int\limits_{\mathbb{R}_+} a(t)f'(t+x)\nu(dt)$. So,

$$\begin{aligned}F(x)&=\int\limits_G a(t)f(t+x)\nu(dt)+\int_{\mathbb{R}_+\setminus G} a(t)f(t+x)\nu(dt)\\&\leq\int\limits_G a(t)f(t+x)\nu(dt)+\frac{F(x)}{2}.\end{aligned}$$

Therefore,

$$F(x)\leq 2\int\limits_G a(t)f(t+x)\nu(dt)\leq 2\mu(x,F)\nu(G)\leq\mu(x,F)\nu_0(2g'(x)),\ x\geq x_0, \tag{2.10}$$

where $\nu_0(t) := \nu\{u \geq 0\colon (\ln f(u))' \leq t\}$.

Now we need the following lemma ([6, 5]).

Lemma 2.1. *For a given non-decreasing function $\nu_0(t)\colon \mathbb{R}_+ \to \mathbb{R}_+$ the condition*

$$(\exists t_0 > 0)\colon \quad \int_{t_0}^{+\infty} \frac{d \ln \nu_0(t)}{t} < +\infty, \tag{2.11}$$

is equivalent to each of the following two conditions: i) condition (2.9); ii) there exists a continuous function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\psi(t) \uparrow +\infty$ $(t_0 \leq t \to +\infty)$ and

$$\int_0^{+\infty} \frac{dt}{\psi(t)} < +\infty, \qquad \ln \nu_0(t) = o(\psi^{-1}(t)) \quad (t \to +\infty). \tag{2.12}$$

For the function $\psi_0(t) = \psi(t)/2$ let us denote a set

$$E := \{x > x_0\colon g'(x) \geq \psi_0(g(x))\}.$$

Then, we obtain the following estimate of the Lebesgue measure for the set E

$$\text{meas}\,(E \cap [x_0, +\infty)) = \int_E dx \leq \int_E \frac{g'(x)}{\psi_0(g(x))} dx \leq \int_0^{+\infty} \frac{dt}{\psi_0(t)} < +\infty, \tag{2.13}$$

i.e., the set E has finite Lebesgue measure. Therefore, from inequality (2.10) and relation (2.12) we obtain finally

$$\begin{aligned} \ln F(x) &\leq \ln 2 + \ln \mu(x, F) + \ln \nu_0(2g'(x)) \\ &\leq \ln 2 + \ln \mu(x, F) + \ln \nu_0(\psi(g(x))) = \ln \mu(x, F) + o(g(x)) \end{aligned}$$

as $x \to +\infty$ $(x \notin E)$. Hence, $\ln F(x) \leq (1 + o(1)) \ln \mu(x, F)$ as $x \to +\infty$ $(x \notin E)$.

□

3. Corollary: Stability of a maximal term

In view of Proposition 2.1 and Lemma 2.1, we obtain the following corollaries.

Corollary 3.1 ([3], Theorem 2)**.** *Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function such that the function $y = \ln f(x)\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a convex function and the right-hand derivative $L(x, f) := (\ln f(x))'_+ \uparrow +\infty$ $(x \geq x_0)$; (λ_n) be a non-negative sequence such that $0 \leq \lambda_n \uparrow +\infty$ $(0 \leq n \uparrow +\infty)$. If a function F represented on $(0, +\infty)$ by a series of form*

$$F(x) = \sum_{n=0}^{\infty} a_n f(\lambda_n + x), \quad a_n \geq 0 \quad (n \geq 0), \tag{3.14}$$

and the condition

$$\sum_{n=1}^{+\infty} \frac{1}{nL(\lambda_n, f)} < +\infty \tag{3.15}$$

holds, then there exists a set $E \subset (0, +\infty)$ of finite Lebesgue measure such that asymptotic relation $\ln F(x) = (1 + o(1)) \ln \mu_F(x)$ holds as $x \to +\infty$ $(x \to +\infty,\ x \notin E)$, where

$$\mu_F(x) = \max\{a_n f(\lambda_n + x)\colon n \geq 0\}.$$

Hence, for series of the form (1.7) we obtain the following corollary.

Corollary 3.2 ([3, Corollary 1])**.** *Let (β_n) be a non-decreasing to $+\infty$ sequence and a function G represented by regularly convergent functional series of form (1.7), where g is an entire function. If condition (1.8) satisfies, then the relation $\ln M_G(r) = (1 + o(1)) \ln \mu_G(r)$ holds as $r \to +\infty$ outside a set of finite logarithmic measure, where $\mu_G(r) = \max\{|b_n| M_g(r\beta_n)\colon n \geq 0\}$.*

Let L be a class of positive continuous on $\mathbb{R}_+ := [0, +\infty)$ functions $l(t)$ such that $l(t) \to +\infty$ $(t \to +\infty)$. By L_+ we denote the subclass of L such that $l(t) \uparrow +\infty$ as $t \to +\infty$, and by $\mathcal{W}$ the class of functions $w \in L_+$ such that

$$\int_1^{+\infty} x^{-2} w(x)dx < +\infty.$$

Let us denote by $D_+(\lambda, f)$ the class of functions $F\colon \mathbb{R}_+ \to \mathbb{R}_+$ of form (3.14). For a series $F \in D_+(\lambda, f)$ and any sequence (b_n), $b_n \in \mathbb{R}_+ \setminus \{0\}$ $(n \geq 0)$ we consider

$$B^+(x) = \sum_{n=0}^{+\infty} a_n b_n f(x + \lambda_n), \qquad B^-(x) = \sum_{n=0}^{+\infty} a_n b_n^{-1} f(x + \lambda_n).$$

We call that a series of the form (3.14) (maximal term of the series) is stable if the relations

$$\ln \mu(x, F) = (1 + o(1)) \ln \mu(x, B^+) = (1 + o(1)) \ln \mu(x, B^-) \tag{3.16}$$

hold as $x \to +\infty$ outside some set $E \subset [0, +\infty)$ of the finite Lebesgue measure, i.e., $\operatorname{meas} E := \int_E dx < +\infty$.

For a function $w \in L$ let us denote

$$B_w(x) = \sum_{n=0}^{+\infty} a_n e^{w(\lambda_n)} f(x\lambda_n).$$

Let us denote

$$\nu_0(t) = \nu\{u \geq 0\colon (\ln f(u))' \leq t\}, \quad \nu(G) = \sum\nolimits_{\lambda_n \in G} e^{w(\lambda_n)}$$

for every bounded set $G \in \mathbb{R}_+$.

Theorem 3.4. *Let $F \in D_+(\lambda, f)$. If there exists a function $w \in L_+$ such that $B_w \in D_+(\lambda, f)$, $\ln \nu_0 \in \mathcal{W}$ and inequalities*

$$e^{-w(\lambda_n)} \leq b_n \leq e^{w(\lambda_n)} \quad (n \geq k_1), \tag{3.17}$$

are valid, then there exists a set $E \subset \mathbb{R}_+$ of finite Lebesque measure such that relation

$$\ln \mu(x, F) = (1 + o(1)) \ln \mu(x, B_+) = (1 + o(1)) \ln \mu(x, B_-) \tag{3.18}$$

holds as $x \to +\infty$ $(x \notin E)$.

Proof of Theorem 3.4. Note that relation (3.18) will follow from the fact that

$$\ln \mu(x, F) = (1 + o(1)) \ln \mu(x, B_w) \tag{3.19}$$

as $x \to +\infty$ outside of some set E of finite Lebesgue measure. Let us prove relation (3.19). Let $a(t), b(t)$ be measurable nonnegative functions on $\mathbb{R}_+$ such that $a(\lambda_n) = a_n$, $b(\lambda_n) = e^{w(\lambda_n)}$ and

$$\mu(x, F) = \sup\{a(t)f(t + x) : t \in \mathbb{R}_+\}, \ \ \mu(x, B_w) = \sup\{a(t)b(t)f(t + x) : t \in \mathbb{R}_+\}.$$

It is enough to take that $a(t) = 0$ for $t \notin \{\lambda_n : n \in \mathbb{Z}_+\}$. Then for all $x \in \mathbb{R}$ we get

$$\mu(x, F) \leq \mu(x, B_w) \leq B_w(x) = \sum_{n=0}^{+\infty} a_n b(\lambda_n) f(x + \lambda_n) = \int_{\mathbb{R}_+} a(t) f(t + x)\nu(dt), \tag{3.20}$$

where measure ν is such that $\nu(G) = \sum_{n=0}^{+\infty} b(\lambda_n)\delta_{\lambda_n}(G)$ for each bounded set $G \subset \mathbb{R}_+$ and $\delta_\lambda(G) = 1$ for $\lambda \in G$ and $\delta_\lambda(G) = 0$ for $\lambda \notin G$. Since G is bounded and $\lambda_n \to +\infty$, it means that the finite number of λ_n belongs to the set G, i.e., for all $n \geq n_0(G)$ one has $\delta_{\lambda_n}(G) = 0$. In view of this, the series $\sum_{n=0}^{+\infty} b(\lambda_n)\delta_{\lambda_n}(G)$ reduces to the finite sum $\sum_{n=0}^{n_0} b(\lambda_n)\delta_{\lambda_n}(G)$. It yields σ-additivity of the measure ν.

From the condition $\ln\nu_0 \in \mathcal{W}$ we immediately get that condition (2.9) of Proposition 2.1 is satisfied, because we substitute $w(x) = \ln\nu_0(x)$ in (2.9). Applying Proposition 2.1 to the integral in (3.20), as $x \to +\infty$ $(x \notin E)$, (here a set E is such as in Proposition 2.1) we obtain

$$\ln\mu(x, F) \le \ln\mu(x, B_w) \le (1 + o(1))\ln\mu_*(x),$$

where $\mu_*(x) = \max\{a(t)f(x+t)\colon t \in \mathbb{R}_+\}$. As for the choice of function $a(t)$, we get $\mu_*(x) = \mu(x, F)$ and deduce relation (3.19). □

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Research Article

Approximation characteristics of Stepanets-Orlicz type spaces

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ABSTRACT. In the paper, we introduce spaces $\mathcal{S}_{M,\Phi}$ and show their connection with the well-known Stepanets spaces $\mathcal{S}^p_\Phi$, Orlicz spaces $\mathcal{S}_M$, and others. We obtain exact upper bounds for quantities analogous to trigonometric widths and best n-term approximations of certain compact sets in these spaces.

Keywords: Stepanets spaces, Orlicz spaces, best approximation, best n-term approximation.

2020 Mathematics Subject Classification: 41A25, 41A65.

1. INTRODUCTION

In 2003, Stepanets [12] introduced the spaces $\mathcal{S}^p_\Phi = \mathcal{S}^p_\Phi(\mathscr{X}, \mathscr{Y})$. His approach made it possible to consider the usual problems of approximation theory in general linear spaces, and at the same time, by considering special cases of his constructions, it is possible to obtain quite meaningful results in many well-known spaces. Here, we consider the spaces $\mathcal{S}_{M,\Phi}$, which constructed similarly to the spaces $\mathcal{S}^p_\Phi$ and generated by a certain Orlicz function $M(u)$, $u \geq 0$. In the case when M is a power function, i.e., $M(u) = u^p$, $p \geq 1$, the spaces $\mathcal{S}_{M,\Phi}$ coincide with the spaces $\mathcal{S}^p_\Phi$. However, this construction also makes it possible to obtain, as special cases of spaces $\mathcal{S}_{M,\Phi}$, some other important spaces of the Orlicz type.

2. THE SPACES $\mathcal{S}_{M,\Phi}$

2.1. **Definition of the spaces $\mathcal{S}_{M,\Phi}$.** Let us introduce spaces $\mathcal{S}_{M,\Phi}$ and show their connection with other well-known spaces. In defining them and further in the paper, we will mainly use the symbols and definitions proposed in [12].

Let $\mathscr{X}$ and $\mathscr{Y}$ be some linear spaces of elements x and y, respectively. Suppose that a linear operator Φ is defined on $\mathscr{X}$ acting in $\mathscr{Y}$, and a functional f is defined on some subset $\mathscr{Y}' \subset \mathscr{Y}$. Let $E(\Phi)$ be the range of the operator Φ, and $\mathscr{X}'$ be the preimage of the set $\mathscr{Y}' \subset E(\Phi)$ under the mapping Φ. In this case, on $\mathscr{X}'$ we can define the functional f' using the equality

$$f'(x) = f(\Phi(x)), \qquad x \in \mathscr{X}' \tag{2.1}$$

If we choose as f a functional that defines a norm (or quasi-norm) on $\mathscr{Y}'$, then equality (2.1) will define a similar quantity on $\mathscr{X}'$. These considerations form the basis for further constructions.

Let $(\mathbb{R}^d, \mathfrak{B}, \mathrm{d}\mu)$, $d \geq 1$, be a d-dimensional Euclidean space of points $\mathbf{t} = (t_1, \ldots, t_d)$, defined on a Borel σ-algebra $\mathfrak{B}$ of subsets of $\mathbb{R}^d$, with a finite σ-additive continuous measure $\mathrm{d}\mu$. Let

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$\mathbb{A}$ be a μ-dimensional subset of $(\mathbb{R}^d, \mathfrak{B}, \mathrm{d}\mu)$, whose μ-measure is equal to a, where a is a finite number, or $a = +\infty$:

$$\mu(\mathbb{A}) = |\mathbb{A}|_\mu = a, \quad a \in (0, +\infty].$$

Let also $\mathscr{Y} = \mathscr{Y}(\mathbb{A}, \mathrm{d}\mu)$ be the set of all functions $f = f(\mathbf{t})$ defined on $\mathbb{A}$ and measurable with respect to the measure $\mathrm{d}\mu$.

Next, let $M(u)$, $u \geq 0$, be an arbitrary Orlicz function, i.e., a non-decreasing convex function such that $M(0) = 0$ and $M(u) \to +\infty$ as $u \to +\infty$. Denote by $L_M(\mathbb{A}, \mathrm{d}\mu)$ the set of all functions $f \in \mathscr{Y}(\mathbb{A}, \mathrm{d}\mu)$ that satisfy the condition

$$(\forall C > 0) \qquad \int_{\mathbb{A}} M(C|f(\mathbf{t})|)\, \mathrm{d}\mu(\mathbf{t}) < +\infty.$$

The linear space $L_M(\mathbb{A}, \mathrm{d}\mu)$ is a Banach space with the Luxembourg norm

$$\|f\|_{L_M(\mathbb{A},\, \mathrm{d}\mu)} := \inf \left\{ \alpha > 0 : \int_{\mathbb{A}} M\Big(\tfrac{|f(\mathbf{t})|}{\alpha}\Big)\, \mathrm{d}\mu(\mathbf{t}) \leq 1 \right\}, \tag{2.2}$$

and it is called the Orlicz space.

Note that in the case when $M(u) = u^p$, $p \geq 1$, the spaces $L_M(\mathbb{A}, \mathrm{d}\mu)$ coincide with the well-known Lebesgue spaces $L_p(\mathbb{A}, \mathrm{d}\mu)$, which consist of functions $f \in \mathscr{Y}(\mathbb{A}, \mathrm{d}\mu)$ such that

$$\|f\|_{L_p(\mathbb{A},\, \mathrm{d}\mu)} = \Big(\int_{\mathbb{A}} |f(\mathbf{t})|^p \mathrm{d}\mu(\mathbf{t}) \Big)^{\frac{1}{p}} < +\infty.$$

Let $\mathscr{X}$ be a linear space of elements x, and Φ be a linear operator acting from $\mathscr{X}$ to $\mathscr{Y}(A, \mathrm{d}\mu)$:

$$\Phi : \mathscr{X} \to \mathscr{Y}(\mathbb{A},\, \mathrm{d}\mu), \quad \Phi(x) =: \widehat{x}, \quad x \in \mathscr{X}, \quad \widehat{x} \in \mathscr{Y}(\mathbb{A},\, \mathrm{d}\mu). \tag{2.3}$$

We set

$$\mathcal{S}_{M,\Phi} = \mathcal{S}_{M,\Phi}(\mathscr{X}, \mathscr{Y}) = \Big\{ x \in \mathscr{X} : \|\widehat{x}\|_{L_M(\mathbb{A},\, \mathrm{d}\mu)} < +\infty \Big\}. \tag{2.4}$$

Thus, the set $\mathcal{S}_{M,\Phi}$ is the set of all elements $x \in \mathscr{X}$ that are preimages of functions from the set $L_M(\mathbb{A},\, \mathrm{d}\mu)$ under the mapping Φ.

Elements $x_1, x_2 \in \mathscr{X}$ are considered identical in $\mathcal{S}_{M,\Phi}$ if, with respect to measure $\mathrm{d}\mu$, $\widehat{x}_1(\mathbf{t}) = \widehat{x}_2(\mathbf{t})$ almost everywhere on $\mathbb{A}$. For elements $x_1, x_2 \in \mathcal{S}_{M,\Phi}$, the Φ-distance between them is defined by the equality

$$\rho_\Phi(x_1, x_2)_{M,\Phi} = \|\Phi(x_1 - x_2)\|_{L_M(\mathbb{A},\, \mathrm{d}\mu)}.$$

An element θ is called a zero element of the set $\mathcal{S}_{M,\Phi}$ if the equality $\widehat{\theta}(\mathbf{t}) = 0$ holds almost everywhere on $\mathbb{A}$.

The distance $\rho_\Phi(\theta, x)_{M,\Phi}$ is called the Φ-norm of the element $x \in \mathcal{S}_{M,\Phi}$ and it is denoted by $\|x\|_{M,\Phi}$. Thus, by definition

$$\|x\|_{M,\Phi} = \rho_\Phi(\theta, x)_{M,\Phi} = \|\widehat{x}\|_{L_M(\mathbb{A},\, \mathrm{d}\mu)}. \tag{2.5}$$

In the terminology accepted in the theory of integral transformations, the element $\widehat{x} = \Phi(x)$ is the image (Φ-image) of an element x, and the set $E(\Phi)$ of values of the operator Φ is the set of images. Thus, the Φ-distance and Φ-norm are the distance and norm in the space of images.

2.2. **Examples of the spaces $\mathcal{S}_{M,\Phi}$.** Let us consider several examples of the simplest implementations of the constructions under consideration. Here, we will say that a certain space $\mathfrak{N}$ is a special case of the space $\mathcal{S}_{M,\Phi}$ if it can be obtained by appropriately choosing the space $\mathscr{X}$, the measure $\mathrm{d}\mu$, and the operator Φ.

2.2.1. *Spaces* $\mathcal{S}^p_\Phi$. In the case when $M(u) = t^p$, $p \geq 1$, the spaces $\mathcal{S}_{M,\Phi}$ coincide with the Stepanets spaces $\mathcal{S}^p_\Phi$ [12]. Note that in [12], many special cases in the above sense of spaces $\mathcal{S}^p_\Phi$ were also presented. Approximative characteristics of the spaces $\mathcal{S}^p_\Phi$ were studied in [12, 13, 16, 8], etc.

2.2.2. *Spaces* $\mathcal{S}_{M,\varphi}$. Let $\mathscr{X}$ be a linear complex space, let $\varphi = \{\varphi_k\}_{k=1}^\infty$ be a fixed countable linearly independent system in this space, and let any pair $x, y \in \mathscr{X}$, in which at least one element belongs to φ, be associated with a number (x, y) such that the following conditions are satisfied:

1) $(x, y) = \overline{(y, x)}$, where $\overline{z}$ is the complex conjugate of z;
2) $(\lambda x_1 + \mu x_2, y) = \lambda(x_1, y) + \mu(x_2, y)$, where λ and μ are arbitrary numbers;
3) $(\varphi_k, \varphi_l) = \begin{cases} 0, & k \neq l; \\ 1, & k = l. \end{cases}$

Thus, the scalar product of elements of the space $\mathscr{X}$ by elements of the system φ is defined. Every element $x \in \mathscr{X}$ is associated with a system of numbers $\widehat{x}_\varphi$ such that

$$\widehat{x}_\varphi(k) = (x, \varphi_k), \qquad k = 1, 2, \ldots \qquad (k \in \mathbb{N})$$

and for arbitrary Orlicz function M, we consider the sets

$$\mathcal{S}_{M,\varphi} = \mathcal{S}_{M,\varphi}(\mathscr{X}) = \Big\{ x \in \mathscr{X} : \sum_{k=1}^\infty M\big(C\widehat{x}_\varphi(k)\big) < \infty, \quad \forall C > 0 \Big\}. \tag{2.6}$$

Elements $x, y \in \mathscr{X}$ are considered identical in $\mathcal{S}_{M,\varphi}$ if $\widehat{x}_\varphi(k) = \widehat{y}_\varphi(k)$ for all $k \in \mathbb{N}$.

For $x \in \mathcal{S}_{M,\varphi}$, the φ-norm of the element x is denoted by $\|x\|_{p,\varphi}$. Thus,

$$\|x\|_{M,\varphi} := \inf \Big\{ \alpha > 0 : \sum_{k=1}^\infty M\left(\tfrac{|\widehat{x}_\varphi(k)|}{\alpha}\right) \leq 1 \Big\}. \tag{2.7}$$

The spaces $\mathcal{S}_{M,\varphi}$ are a special case of the spaces $\mathcal{S}_{M,\Phi}$. Indeed, in the given space $\mathscr{X}$, we define an operator Φ that for every $x \in \mathscr{X}$ assigns the sequence $y = \{\widehat{x}_k\}_{k=1}^\infty$. We take the space $\mathbb{R}^1$ with measure $\mathrm{d}\mu$, whose support is the set $\mathbb{Z}^1$ of integer points k in which $\mu(k) \equiv 1$, and set $\mathbb{A} = \big\{ k \in \mathbb{Z}^1, \quad k \geq 1 \big\}$. In this case $\mathscr{Y}(\mathbb{A}, \mathrm{d}\mu)$ is the set of all sequences y, and the functional (2.2) has the form

$$\|y\|_{L_M(\mathbb{A},\mathrm{d}\mu)} = \inf \Big\{ \alpha > 0 : \sum_{k=1}^\infty M\big(\tfrac{|y_k|}{\alpha}\big) \leq 1 \Big\}. \tag{2.8}$$

2.2.3. *Spaces* $\mathcal{S}^\mu_{M,\varphi}$. These spaces are defined in the same way as the spaces S^p_φ, except that the functionals of the form

$$\sum_{k=1}^\infty M(\cdot)$$

in the equalities corresponding to equalities (2.6)–(2.8), are replaced by functionals of the form

$$\sum_{k=1}^\infty \mu_k M(\cdot),$$

where $\mu = \{\mu_k\}_{k=1}^\infty$ is a given system of non-negative numbers, $\mu_k \geq 0$, $k \in \mathbb{N}$. In this case, if $\mu_k \equiv 1$, then $\mathcal{S}^\mu_{M,\varphi} = \mathcal{S}_{M,\varphi}$.

It is clear that these spaces are a special case of the spaces $S_{M,\Phi}$. In this case, we similarly take the space $\mathbb{R}^1$ with measure $\mathrm{d}\mu$, whose support is the set $\mathbb{Z}^1$ of integer points k in which $\mu(k) = \mu_k$ and $\mathbb{A} = \{k \in \mathbb{Z}^1, \ k \geq 1\}$.

2.2.4. *Spaces* $\mathcal{S}_M(\mathbb{T}^d) = \mathcal{S}_{M,\mathcal{F}}(L_1(\mathbb{T}^d))$. Let, as above, $\mathbb{R}^d$ be an d-dimensional, $d \geq 1$, Euclidean space of vectors $\mathbf{t} = (t_1, \ldots, t_d)$ and $\mathbb{Z}^d$ be an integer lattice in $\mathbb{R}^d$. Denote by $\mathbb{T}^d := [0, 2\pi)^d$ a d-dimensional torus and set $(\mathbf{t}, \mathbf{y}) := t_1 y_1 + \ldots + t_d y_d$, $\mathbf{t}, \mathbf{y} \in \mathbb{R}^d$

Let, further, $L_1 = L_1(\mathbb{T}^d)$ be the set of all Lebesgue-summable on $\mathbb{R}^d$ functions $f(\mathbf{t}) = f(t_1, \ldots t_d)$, 2π-periodic in each variable. We take as $\mathscr{X}$ the space $L_1(\mathbb{T}^d)$ and define on it the operator Φ (which we will denote by $\mathcal{F}$), setting

$$\mathcal{F}(f) = (2\pi)^{-d} \int\limits_{\mathbb{T}^d} f(\mathbf{t})\, e^{-\mathrm{i}(\mathbf{k},\mathbf{t})} \mathrm{d}\mathbf{t} = \widehat{f}(\mathbf{k}), \quad \mathbf{k} \in \mathbb{Z}^m.$$

This operator maps the space $L_1(\mathbb{T}^d)$ to the set $\mathscr{Y}$ of functions y defined on the integer lattice $\mathbb{Z}^d$. Let $\mathrm{d}\mu$ be a measure in the space $\mathbb{R}^d$, whose support is the set $\mathbb{Z}^d$, where it equals one. In this case, the functional (2.2) has the form

$$\|f\|_{L_M(\mathbb{R}^d,\, \mathrm{d}\mu)} := \inf \left\{ \alpha > 0 : \int\limits_{\mathbb{R}^d} M\Big(\tfrac{|f(\mathbf{t})|}{\alpha}\Big)\, \mathrm{d}\mu(\mathbf{t}) \leq 1 \right\} = \inf \left\{ \alpha > 0 : \sum_{k \in \mathbb{Z}^d} M\Big(\tfrac{|\widehat{f}(\mathbf{k})|}{\alpha}\Big) \leq 1 \right\},$$

and the space $\mathcal{S}_{M,\Phi}$ (which we denote by $\mathcal{S}_M(\mathbb{T}^d)$) is defined by the relation

$$\mathcal{S}_M\left(\mathbb{T}^d\right) = \mathcal{S}_{M,\mathcal{F}}(L_1(\mathbb{T}^d)) = \left\{ f \in L_1(\mathbb{T}^d) : \quad (\forall C > 0) \sum_{k \in \mathbb{Z}^d} M\big(C|\widehat{f}(\mathbf{k})|\big) < +\infty \right\}.$$

Note that the spaces $\mathcal{S}_M(\mathbb{T}^1)$ were considered in [1, 2], where, in particular, direct and inverse approximation theorems in these spaces were proved. These spaces coincide with the spaces $S_{M,\varphi}(L)$ generated by the set $L_1(\mathbb{T}^d)$ and the system $\varphi = \{\tau_s\}_{s=1}^{\infty}$, $\tau_s = \tau_s(\mathbf{x}) = \{\mathrm{e}^{\mathrm{i}(\mathbf{k}_s,\mathbf{x})}\}$, $\mathbf{k}_s \in \mathbb{Z}^d$, $s \in \mathbb{N}$, which can be obtained by arbitrarily numbering the elements of the system $\{\mathrm{e}^{\mathrm{i}(\mathbf{k},\mathbf{x})}\}_{\mathbf{k} \in \mathbb{Z}^d}$.

In the case when $M(u) = u^p$, $p \geq 1$, the spaces $\mathcal{S}_{M,\varphi}$ are the spaces $\mathcal{S}^p_{\varphi}$, the spaces $\mathcal{S}^{\mu}_{M,\varphi}$ are the spaces $\mathcal{S}^{p,\mu}_{\varphi}$, and the spaces $\mathcal{S}_M(\mathbb{T}^d)$ are the Wiener type spaces $\mathcal{S}^p(\mathbb{T}^d)$ [11], which were studied by many authors (see, for example [5, 15, 14, 13, 6, 7], and references therein).

3. Multipliers. Approximating aggregates and objects of approximation

Approximating aggregates for elements $x \in \mathcal{S}_{M,\Phi}$ and objects of approximation are defined similarly to [12]. We use elements from $\mathcal{S}_{M,\Phi}$ whose images have supports γ_σ of given measure σ. It is clear that exactly this principle is used in the classical case in the construction of, e.g., trigonometric polynomials for the approximation of a given periodic function if the operator Φ is understood as the mapping of functions into the set of their Fourier coefficients. In the general case, there arise certain problems related to the fact that the spaces $\mathcal{S}_{M,\Phi}$ can be incomplete. In this connection, we introduce the following definitions.

Let $\omega = \omega(\mathbf{t})$ be a certain function from $\mathscr{Y}(\mathbb{A}, \mathrm{d}\mu)$. Then we denote by $\mathcal{M}^{\omega}_{\Phi}$ the operator acting from $\mathscr{X}$ into $\mathscr{X}$ that associates $x \in \mathscr{X}$ with an element $x_\omega \in \mathscr{X}$ such that if $\Phi(x) = \widehat{x}$, then $\widehat{x}_\omega(\mathbf{t}) = \Phi(x_\omega) = \omega(\mathbf{t})\widehat{x}(\mathbf{t})$ almost everywhere on $\mathbb{A}$. The operator $\mathcal{M}^{\omega}_{\Phi}$ is called the multiplier of the operator Φ generated by the function ω. Let $\Omega_\Phi(\mathscr{X}) = \Omega_\Phi(\mathscr{X}, \mathscr{X})$ denote the subset of functions ω from $\mathscr{Y}(\mathbb{A}, \mathrm{d}\mu)$ for which the multipliers $\mathcal{M}^{\omega}_{\Phi}$ exist.

If $\mathfrak{N}$ and $\mathfrak{N}'$ are some subsets of $\mathscr{X}$, $\omega \in \Omega_\Phi(\mathscr{X})$, and the operator $\mathcal{M}^{\omega}_{\Phi}$ maps $\mathfrak{N}$ into $\mathfrak{N}'$, then we say that $\mathcal{M}^{\omega}_{\Phi}$ has the type $(\mathfrak{N}, \mathfrak{N}')$. In particular, if $\mathcal{M}^{\omega}_{\Phi}$ maps $\mathcal{S}_{M,\Phi}$ into $\mathcal{S}_{M,\Phi}$, then the operator $\mathcal{M}^{\omega}_{\Phi}$ has the type $(\mathcal{S}_{M,\Phi}, \mathcal{S}_{M,\Phi})$, or, briefly, the type (M, M). The set of functions ω generating operators of the type (M, M) is denoted by $\Omega_{M,\Phi}$. Thus, if $\omega \in \Omega_{M,\Phi}$ and the operator $\mathcal{M}_\Phi^\omega$ acts from $\mathcal{S}_{M,\Phi}$, then it acts into $\mathcal{S}_{M,\Phi}$. In this case, every $x \in \mathcal{S}_{M,\Phi}$ is associated with an element $x_\omega = \mathcal{M}_\Phi^\omega(x)$ for which the following equality holds almost everywhere on $\mathbb{A}$:

$$\widehat{x}_\omega(\mathbf{t}) = \Phi(x_\omega) = \omega(\mathbf{t})\widehat{x}(\mathbf{t}), \quad \widehat{x}_\omega \in L_M(\mathbb{A}, \mathrm{d}\mu). \tag{3.9}$$

Given $\sigma > 0$, assume that γ_σ is a μ-measurable set in $\mathbb{A}$, $\mathrm{mes}_\mu \gamma_\sigma =: |\gamma_\sigma| = \sigma$, $\sigma \le a$, and $\lambda = \lambda(\mathbf{t})$ is a measurable function with support γ_σ. Also assume that, for a given Orlicz function M, we have $\lambda \in \Omega_{M,\Phi}$ and $U_{\gamma_\sigma}(x, \lambda) := x_\lambda = \mathcal{M}_\Phi^\lambda(x)$. According to (3.9), we get

$$\widehat{U}_{\gamma_\sigma}(x,\lambda) = \Phi(U_{\gamma_\sigma}(x,\lambda)) = \begin{cases} \lambda(\mathbf{t})\widehat{x}(\mathbf{t}), & \mathbf{t} \in \gamma_\sigma, \\ 0, & \mathbf{t}\bar{\in}\gamma_\sigma, \end{cases} \qquad x \in \mathcal{S}_{M,\Phi}. \tag{3.10}$$

The elements $U_{\gamma_\sigma}(x, \lambda)$ are considered as approximating aggregates for $x \in \mathcal{S}_{M,\Phi}$. In this case, if $\lambda(\mathbf{t}) \equiv 1$ on γ_σ, i.e., if $\lambda(\mathbf{t})$ coincides with the characteristic function $\chi_{\gamma_\sigma}(\mathbf{t})$ of the set γ_σ, then we set $U_{\gamma_\sigma}(x, \chi_{\gamma_\sigma}) =: U_{\gamma_\sigma}(x)$.

Let $\Gamma_\sigma = \Gamma_\sigma(\mathbb{A})$ be the set of all measurable subsets of $\mathbb{A}$ whose measures are equal to σ. We say that, for a given Orlicz function M, an operator Φ satisfies condition (A_M) if the functions $\chi_{\gamma_\sigma}(\mathbf{t})$ of all sets $\gamma_\sigma \in \Gamma_\sigma$ belong to $\Omega_{M,\Phi}$ for all $\sigma \in [0, a)$. Thus, if Φ satisfies condition (A_M), then all elements $U_{\gamma_\sigma}(x)$ are defined for any $x \in \mathcal{S}_{M,\Phi}$ and are contained in $\mathcal{S}_{M,\Phi}$. The element $U_{\gamma_\sigma}(x)$ is called the restriction of an element x of rank σ, and the element $U_{\gamma_\sigma}(x, \lambda)$ is called the λ-restriction of x of the rank σ.

Let M be any Orlicz function and let $x \in \mathcal{S}_{M,\Phi}$. Then, by virtue of (2.5) and (3.10), we get

$$\begin{aligned} &\|x - U_{\gamma_\sigma}(x,\lambda)\|_{M,\Phi} \\ =&\|\widehat{x} - \widehat{U}_{\gamma_\sigma}(x,\lambda)\|_{L_M(\mathbb{A},\mathrm{d}\mu)} \\ =&\inf\left\{\alpha > 0 : \int_{\gamma_\sigma} M\Big(\tfrac{|1-\lambda(\mathbf{t})||\widehat{x}(\mathbf{t})|}{\alpha}\Big)\, d\mu(\mathbf{t}) + \int_{\mathbb{A}\setminus\gamma_\sigma} M\Big(\tfrac{|\widehat{x}(\mathbf{t})|}{\alpha}\Big)\mathrm{d}\mu(\mathbf{t}) \le 1\right\}. \end{aligned}$$

Hence, we arrive at the following statement:

Proposition 3.1. *Suppose that M is any Orlicz function, $x \in \mathcal{S}_{M,\Phi} = \mathcal{S}_{M,\Phi}(\mathscr{X}, \mathscr{Y})$, $\gamma_\sigma \in \Gamma_\sigma$ and the operator Φ satisfies condition (A_M). Then*

$$\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi} := \inf_{\lambda \in \Omega_{M,\Phi}} \| x - U_{\gamma_\sigma}(x,\lambda)\|_{M,\Phi} = \| x - U_{\gamma_\sigma}(x)\|_{M,\Phi}.$$

Furthermore, the following equality is true:

$$\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi} = \inf\left\{\alpha > 0 : \int_{\mathbb{A}\setminus\gamma_\sigma} M\Big(\tfrac{|\widehat{x}(\mathbf{t})|}{\alpha}\Big)\mathrm{d}\mu(\mathbf{t}) \le 1\right\}. \tag{3.11}$$

Thus, if $\chi_{\gamma_\sigma} \in \Omega_{M,\Phi}$, among all elements $U_{\gamma_\sigma}(x, \lambda)$ generated by the multipliers $\mathcal{M}_\Phi^\lambda$ and satisfying condition (3.10), the element $U_{\gamma_\sigma}(x)$ has the least deviation from an element x in Φ-norm in the space $\mathcal{S}_{M,\Phi}$, i.e., among all λ-restrictions of x of given rank σ, its restriction for $\lambda(\mathbf{t}) \equiv 1$ is the closest one to x. It is clear that this property is an analog of the minimum property of Fourier sums in the Hilbert spaces.

Let $\Gamma = \{\gamma_\sigma\}_{\sigma>0}$, $|\gamma_\sigma| = \sigma$, be a family of measurable subsets of $\mathbb{A}$ that exhausts the entire set $\mathbb{A}$ for $\sigma \to +\infty$, i.e., it possesses the property that any point $\mathbf{t} \in \mathbb{A}$ is contained in all sets γ_σ for all sufficiently large values of σ and, therefore, for any $\alpha > 0$,

$$\lim_{\sigma\to+\infty} \int_{\gamma_\sigma} M\Big(\tfrac{|\widehat{x}(\mathbf{t})|}{\alpha}\Big)\mathrm{d}\mu(\mathbf{t}) = \int_{\mathbb{A}} M\Big(\tfrac{|\widehat{x}(\mathbf{t})|}{\alpha}\Big)\mathrm{d}\mu(\mathbf{t}), \qquad \forall x \in \mathcal{S}_{M,\Phi}. \tag{3.12}$$

Combining relations (3.11) and (3.12), we see that

$$\lim_{\substack{\sigma\to+\infty\\ \gamma_\sigma\in\Gamma}} \mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi} = 0, \qquad \forall x\in\mathcal{S}_{M,\Phi}.$$

The object of approximation are the classes of Ψ-integrals of all elements belonging to the unit balls $U_{M,\Phi}$ or U^p_Φ of the spaces $\mathcal{S}_{M,\Phi}$ and S^p_Φ, respectively, under conditions that guarantee the embedding $U^p_\Phi\subset\mathcal{S}_{M,\Phi}$. The concept of Ψ-integral is introduced as follows [12]. Let $\Psi=\Psi(\mathbf{t})$ be an arbitrary function from $\Omega_\Phi(\mathscr{X})$ and let M^Ψ_Φ be the multiplier of an operator Φ generated by this function. In this case, the image x_Ψ of an element x under the mapping M^Ψ_Φ is called the Ψ-integral of an element x and is denoted by $M^\Psi_\Phi(x)=x_\Psi=\mathcal{J}^\Psi x$. In certain cases, it is convenient to call x the Ψ-derivative of x_Ψ and write $x=D^\Psi x_\Psi$. Thus, if x_Ψ is the Ψ-integral of x, then almost everywhere on $\mathbb{A}$, we have

$$\widehat{x}_\Psi=\Phi(\mathcal{J}^\Psi x)=\Psi(\mathbf{t})\widehat{x}(\mathbf{t}). \tag{3.13}$$

Let $\mathfrak{N}$ be a certain subset of $\mathscr{X}$. By $\Psi\,\mathfrak{N}$, we denote the set of Ψ-integrals of all $x\in\mathfrak{N}$ for which they exist. In particular, if

$$U_{M,\Phi}=\{x\in\mathcal{S}_{M,\Phi}:\|x\|_{M,\Phi}\le 1\} \qquad\text{and}\qquad U^p_\Phi=\{x\in\mathcal{S}^p_\Phi:\|x\|_{p,\Phi}\le 1\}$$

are the unit balls in certain spaces $\mathcal{S}_{M,\Phi}$ and S^p_Φ, $p>0$, then $\Psi U_{M,\Phi}$ and ΨU^p_Φ are the sets of Ψ-integrals of all $x\in U_{M,\Phi}$ and $x\in U^p_\Phi$ for which these integrals exist.

Comparing relations (3.13) and (3.9), we conclude that, as functions Ψ for which the definition of Ψ-integral is correct, one can choose any function from $\Omega_{M,\Phi}$. In this case, the inclusions $\Psi\mathcal{S}_{M,\Phi}\subset\mathcal{S}_{M,\Phi}$ and $\Psi\mathcal{S}^p_\Phi\subset\mathcal{S}_{M,\Phi}$ are valid.

4. Approximation characteristics

Let γ_σ, $\sigma\in(0,a)$, be a fixed set of Γ_σ. Consider the following quantities:

$$\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi}=\inf_{\lambda\in\Omega_{M,\Phi}}\|x-U_{\gamma_\sigma}(x,\lambda)\|_{M,\Phi}, \qquad x\in\mathcal{S}_{M,\Phi}, \tag{4.14}$$

$$\mathscr{E}_{\gamma_\sigma}(\Psi U_{M,\Phi})_{M,\Phi}=\sup_{x\in\Psi U_{M,\Phi}}\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi} \tag{4.15}$$

and

$$\mathscr{D}_\sigma(\Psi U_{M,\Phi})_{M,\Phi}=\inf_{\gamma_\sigma\in\Gamma_\sigma}\mathscr{E}_{\gamma_\sigma}(\Psi U_{M,\Phi})_{M,\Phi}. \tag{4.16}$$

In the case of approximating periodic functions with trigonometric polynomials, the quantity $\mathscr{E}_{\gamma_\sigma}(x)_p$ corresponds to the best approximation of the function x using polynomials of degree σ matching the set γ_σ, the quantity $\mathscr{E}_{\gamma_\sigma}(\Psi U_{M,\Phi})_{M,\Phi}$ corresponds to the exact upper bound on a given set of functions of such best approximations. The quantity $\mathscr{D}_\sigma(\Psi U_{M,\Phi})_{M,\Phi}$ resembles the trigonometric width of order σ of the set $\Psi U_{M,\Phi}$.

We also consider the following characteristics, which, in the periodic case, correspond to quantities related to the best σ-term approximation:

$$e_\sigma(x)_{M,\Phi}=\inf_{\gamma_\sigma\in\Gamma_\sigma}\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi}, \qquad x\in\mathcal{S}_{M,\Phi}. \tag{4.17}$$

Denote $\Psi U^p_{M,\Phi}=\Psi U^p_\Phi\cap\mathcal{S}_{M,\Phi}$, $0<p<\infty$, and

$$e_\sigma(\Psi U^p_{M,\Phi})_{M,\Phi}=\sup_{x\in\Psi U^p_{M,\Phi}}e_\sigma(x)_{M,\Phi}. \tag{4.18}$$

In what follows, we restrict ourselves to the case when corresponding characteristic functions $\chi_{\gamma_\sigma}(\cdot)$ belong to $\Omega_{M,\Phi}$, i.e., the operator Φ satisfies condition (A_M). In this case, according to Proposition 3.1, of major interest are quantities (4.14)–(4.18), where $\lambda(\mathbf{t}) = \chi_{\gamma_\sigma}(\mathbf{t})$. In this connection, we set

$$\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi} = \| x - U_{\gamma_\sigma}(x)\|_{M,\Phi}, \qquad x \in \mathcal{S}_{M,\Phi}, \tag{4.19}$$

$$\mathscr{E}_{\gamma_\sigma}(\Psi U_{M,\Phi})_{M,\Phi} = \sup_{x\in\Psi U_{M,\Phi}} \mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi} \tag{4.20}$$

and

$$\mathscr{D}_\sigma(\Psi U_{M,\Phi})_{M,\Phi} = \inf_{\gamma_\sigma\in\Gamma_\sigma} \mathscr{E}_{\gamma_\sigma}(\Psi U_{M,\Phi})_{M,\Phi}.$$

Similarly,

$$e_\sigma(x)_{M,\Phi} = \inf_{\gamma_\sigma\in\Gamma_\sigma} \| x - U_{\gamma_\sigma}(x)\|_{M,\Phi} \tag{4.21}$$

and

$$e_\sigma(\Psi U^p_{M,\Phi})_{M,\Phi} = \sup_{x\in\Psi U^p_{M,\Phi}} e_\sigma(x)_{M,\Phi}. \tag{4.22}$$

Theorem 4.1. *Let M be an arbitrary Orlicz function, and let $\Psi = \Psi(\mathbf{t})$ be an arbitrary function from the set $\mathscr{Y}(\mathbb{A}, \mathrm{d}\mu)$, essentially bounded on $\mathbb{A}$, i.e.,*

$$\operatorname*{ess\,sup}_{\mathbf{t}\in A} |\Psi(\mathbf{t})| < \infty \tag{4.23}$$

and in the case where the set $\mathbb{A}$ is unbounded, let

$$\lim_{|\mathbf{t}|\to+\infty} \Psi(\mathbf{t}) = 0. \tag{4.24}$$

Further, let $\mathscr{X}$ be an arbitrary linear space, and Φ be an operator that acts according to (2.3) and satisfies the condition (A_M). Then, for any $\gamma_\sigma \in \Gamma_\sigma$, $\sigma < a$, the following estimates hold:

$$\mathscr{E}_{\gamma_\sigma}(\Psi U_{M,\Phi})_{M,\Phi} \le \bar{\Psi}_{\gamma_\sigma}(0+), \tag{4.25}$$

where $\bar{\Psi}_{\gamma_\sigma}$ is the decreasing rearrangement of the function

$$\Psi_{\gamma_\sigma}(\mathbf{t}) = \begin{cases} |\Psi(\mathbf{t})|, & \mathbf{t} \in \mathbb{A} \setminus \gamma_\sigma, \\ 0, & \mathbf{t} \in \gamma_\sigma, \end{cases} \tag{4.26}$$

and

$$\mathscr{D}_\sigma(\Psi U_{M,\Phi})_{M,\Phi} \le \bar{\Psi}(\sigma+), \tag{4.27}$$

where $\bar{\Psi}$ is the decreasing rearrangement of the function $|\Psi(\mathbf{t})|$, $\mathbf{t} \in \mathbb{A}$.

If, in addition, for any $\gamma_\sigma \in \Gamma_\sigma$, $\sigma \in (0, a)$, the characteristic functions χ_{γ_σ} belong to the set $E(\Phi)$ of values of the operator Φ, and their preimages U_{γ_σ} have Ψ-integrals, then relations (4.25) and (4.27) are equalities. In this case, there exists a set γ^*_σ in Γ_σ for which the following equalities hold:

$$\mathscr{E}_{\gamma^*_\sigma}(\Psi U_{M,\Phi})_{M,\Phi} = \mathscr{D}_\sigma(\Psi U_{M,\Phi})_{M,\Phi} = \bar{\Psi}(\sigma+). \tag{4.28}$$

In particular, any measurable subset of the set $\{\mathbf{t} \in \mathbb{A} : |\Psi(\mathbf{t})| \ge \bar{\Psi}(\sigma+)\}$ that contains the set $\{\mathbf{t} \in \mathbb{A} : |\Psi(\mathbf{t})| > \bar{\Psi}(\sigma+)\}$ can be taken as γ^*_σ.

Note that conditions (4.23) and (4.24) guarantee that for the function $|\Psi(\mathbf{t})|$, its distribution function $m_{|\Psi|}(y)$,

$$m_{|\Psi|}(y) = \mathrm{mes}_\mu E_y, \qquad E_y = \{\mathbf{t} \in \mathbb{A} : |\Psi(\mathbf{t})| \ge y\}, \qquad y \ge 0,$$

takes only finite values from the interval $[0, a]$ for any $y > 0$. Therefore, according to the definition of a decreasing rearrangement (see, for example, [3, Ch. 10], [4, Ch. 6], [12]), the functions $\bar{\Psi}_{\gamma_\sigma}(t)$ and $\bar{\Psi}(t)$ are always defined.

We also note that, in the case where $E(\Phi) = L_M(\mathbb{A}, \mathrm{d}\mu)$, the operator Φ satisfies condition (A_M). Moreover, by virtue of conditions (4.23) and (4.24), the requirements that guarantee the equality in relations (4.25) and (4.27) are also satisfied.

Proof. Let's use the proof scheme of Theorem 1 of [12] and Theorem 2 of [9]. Note that the function $\Psi_{\gamma_\sigma}(\mathbf{t})$ is essentially bounded on $\mathbb{A}\backslash\gamma_\sigma$ by virtue of (4.23). Therefore, its rearrangement in decreasing order is bounded. Hence, there exists the limit

$$\bar{\Psi}_{\gamma_\sigma}(0+) = \lim_{t\to 0+} \bar{\Psi}_{\gamma_\sigma}(t) =: y_\sigma. \tag{4.29}$$

First, we prove relation (4.25). If $x \in \Psi U_{M,\Phi}$, according to (4.19), (2.4), (3.9) and (2.2), we get

$$\begin{aligned}\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi} &= \|\Phi(x - U_{\gamma_\sigma}(x))\|_{L_M(\mathbb{A},\, \mathrm{d}\mu)} \\ &= \|\widehat{x} - \chi_{\gamma_\sigma}\widehat{x}\|_{L_M(\mathbb{A},\, \mathrm{d}\mu)} \\ &= \|\,\Psi y - \chi_{\gamma_\sigma}\Psi y\|_{L_M(\mathbb{A},\, \mathrm{d}\mu)},\end{aligned} \tag{4.30}$$

where y is some function of the unit ball $U_M(\mathbb{A}, \mathrm{d}\mu)$ in the space $L_M(\mathbb{A}, \mathrm{d}\mu)$. Further, we can use Theorem 2 from the paper [9], where, in fact, it is shown that

$$\sup_{y\in U_M(\mathbb{A},\, \mathrm{d}\mu)} \|\Psi y - \chi_{\gamma_\sigma}\Psi y\|_{L_M(\mathbb{A},\, \mathrm{d}\mu)} = \bar{\Psi}_{\gamma_\sigma}(0+). \tag{4.31}$$

Combining (4.20), (4.30) and (4.31), we see that relation (4.25) is true indeed. Considering the lower bounds of both parts of (4.25) over the set Γ_σ, we get

$$\mathscr{D}_\sigma(\Psi U_{M,\Phi})_{M,\Phi} \le \inf_{\gamma_\sigma\in\Gamma_\sigma} \bar{\Psi}_{\gamma_\sigma}(0+). \tag{4.32}$$

In view of relation (4.26), we can conclude that the least value of the quantity $\bar{\Psi}_{\gamma_\sigma}(0+)$ is realized in the case where $\gamma_\sigma = \gamma_\sigma^*$, and this value is equal to $\bar{\Psi}(\sigma+)$, i.e.,

$$\inf_{\gamma_\sigma\in\Gamma_\sigma} \bar{\Psi}_{\gamma_\sigma}(0+) = \bar{\Psi}_{\gamma_\sigma^*}(0+) = \bar{\Psi}(\sigma+). \tag{4.33}$$

This proves relation (4.27).

Now assume that, for any $\gamma_\sigma \in \Gamma_\sigma$, $\sigma \in (0, a)$, the characteristic function χ_{γ_σ} belong to the set $E(\Phi)$ of values of the operator Φ, and its preimage U_{γ_σ} has Ψ-integrals, which belongs to $\mathcal{S}_{M,\Phi}$. For a given $\gamma_\sigma \in \Gamma_\sigma$, we take any number y from the interval $(0, y_\sigma)$. Assume that $e_y = \{\mathbf{t} \in \mathbb{A} \setminus \gamma_\sigma : \varphi_{\gamma_\sigma}(\mathbf{t}) \ge y\}$ and $\bar{e}_y$ is any subset of the set e_y, whose μ-measure does not exceed 1, i.e., $\mathrm{mes}_\mu \bar{e}_y \le 1$. Let

$$h_y(\mathbf{t}) := \chi_{\bar{e}_y}(\mathbf{t}) M^{-1}\Big((\mathrm{mes}_\mu\, \bar{e}_y)^{-1}\Big),$$

where $\chi_{\bar{e}_y}$ is a characteristics function of the set $\bar{e}_y$ and M^{-1} is the inverse function of the function M. Let also U_y be the preimage of the function h_y under the mapping of Φ, i.e., $\Phi(U_y) = h_y$, and $x_\Psi = \mathcal{J}^\Psi U_y$ be the Ψ-integral of the element U_y. By virtue of the above assumptions, all elements constructed exist, and, since

$$\int_{\mathbb{A}} M(h_y)\, \mathrm{d}\mu(\mathbf{t}) \le 1,$$

we get $h_y \in U_M^+(\mathbb{A})$ and $x_\Psi \in \Psi U_{M,\Phi}$. For the element x_Ψ, relations (4.30) and (4.25) yields

$$\begin{aligned}\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi} &= \|\Phi(x_\Psi - U_{\gamma_\sigma}(x_\Psi))\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)}\\ &= \|\Psi h_y\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)}\\ &= \inf\left\{\alpha>0\ :\ \int_{\mathbb{A}\setminus\gamma_\sigma} M\big(\Psi_{\gamma_\sigma}(\mathbf{t})h_y(\mathbf{t})/\alpha\big)\,d\mu\le 1\right\}\\ &\ge y.\end{aligned}$$

Taking into account the arbitrariness of the choice of y from the interval $(0, y_\sigma)$, we can conclude that the set $\Psi U_{M,\Phi}$ contains elements x for which the values of $\mathscr{E}_{\gamma_\sigma}(x)_{M,\Phi}$ are arbitrarily close to the value of y_{γ_σ}. With regard for relation (4.29), this means that, in the case considered, relation (4.25) is, in fact, the equality. Then, according to (4.32) and (4.33), relation (4.27) is also the equality. If, in this case, if γ_σ^* is any measurable subset of the set $\{\mathbf{t}\in\mathbb{A}:\ |\Psi(\mathbf{t})|\ge\bar\Psi(\sigma+)\}$ that contains the set $\{\mathbf{t}\in\mathbb{A}:\ |\Psi(\mathbf{t})|>\bar\Psi(\sigma+)\}$, then

$$\bar\Psi_{\gamma_\sigma^*}(0+) = \bar\Psi(\sigma+).$$

Then, according to relation (4.25) (which is now the equality), we get

$$\mathscr{E}_{\gamma_\sigma^*}(\Psi U_{M,\Phi})_{M,\Phi} = \bar\Psi(\sigma+).$$

This yields (4.28).

□

Note that in the spaces $\mathcal{S}_\Phi^p$, the assertions, similar to Theorem 4.1, were obtained in [12, 16, 8]. In particular, in the case when $M(u)=u^p$, $p\ge 1$, the statement of Theorem 4.1 coincides with the statement of Theorem 1 in [12].

Theorem 4.2. *Let $p\in(0,+\infty)$ and M be an arbitrary Orlicz function such that the function $M(u^{1/p})$ is also Orlicz function. Let also $\Psi=\Psi(\mathbf{t})$ be an arbitrary function from the set $\mathscr{Y}(\mathbb{A},\mathrm{d}\mu)$, essentially bounded on $\mathbb{A}$, which, in the case of an unbounded set $\mathbb{A}$ satisfies the condition (4.24). Further, let $\mathscr{X}$ be an arbitrary linear space, and Φ be an operator that acts according to (2.3) and satisfies the condition (A_M). Then for any σ, $\sigma\in(0,a)$, the following inequality holds:*

$$e_\sigma(\Psi U^p_{M,\Phi})_{M,\Phi} \le \sup_{l\in(\sigma,a]} \frac{\left(\int_0^l \bar\Psi^{-p}(t)dt\right)^{-\frac{1}{p}}}{M^{-1}\left(\frac{1}{l-\sigma}\right)}, \tag{4.34}$$

where M^{-1} is the inverse function of the function M and $\bar\Psi$ is the decreasing rearrangement of the function $|\Psi(\mathbf{t})|$, $\mathbf{t}\in\mathbb{A}$. The value of the exact upper bound in (4.34) is attained at some finite value $l=l^*$.

If, in addition, the set $E(\Phi)$ of values of the operator Φ coincides with the entire space $L_M(\mathbb{A},\mathrm{d}\mu)$, then relation (4.34) is equality.

Proof. Let $x\in\Psi U^p_{M,\Phi}$. By virtue of (4.21), (2.5) and (3.9), we see that

$$\begin{aligned}e_\sigma(x)_{M,\Phi} &= \inf_{\gamma_\sigma\in\Gamma_\sigma}\|\Phi(x-U_{\gamma_\sigma}(x))\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)}\\ &= \inf_{\gamma_\sigma\in\Gamma_\sigma}\|\widehat{x}-\chi_{\gamma_\sigma}\widehat{x}\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)}\\ &= \inf_{\gamma_\sigma\in\Gamma_\sigma}\|\,|\Psi|y-\chi_{\gamma_\sigma}|\Psi|y\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)}\\ &= \inf_{\gamma_\sigma\in\Gamma_\sigma}\inf\left\{\alpha>0:\int_{\mathbb{A}\setminus\gamma_\sigma} M\Big(\tfrac{|\Psi(\mathbf{t})|y(\mathbf{t})}{\alpha}\Big)d\mu(\mathbf{t})\le 1\right\},\end{aligned}$$

where y is a certain function such that the function $|y|$ belongs to the intersection of the space $L_M(\mathbb{A},\, \mathrm{d}\mu)$ and the set $U_p^+(\mathbb{A}, \mathrm{d}\mu)$ of all non-negative functions from the unit ball $U_p(\mathbb{A}, \mathrm{d}\mu)$ of the space $L_p(\mathbb{A}, \mathrm{d}\mu)$, i.e., $|y| \in L_M(\mathbb{A},\, \mathrm{d}\mu) \cap U_p^+(\mathbb{A}, \mathrm{d}\mu) =: \mathcal{U}_{p,M}(\mathbb{A})$. Then by (4.22), we get

$$e_\sigma(\Psi U^p_{M,\Phi})_{M,\Phi} \le \sup_{h\in\mathcal{U}_{p,M}(\mathbb{A})} \inf_{\gamma_\sigma\in\Gamma_\sigma} \inf\left\{\alpha>0: \int_{\mathbb{A}\setminus\gamma_\sigma} M\Big(\tfrac{|\Psi(\mathbf{t})|y(\mathbf{t})}{\alpha}\Big) d\mu(\mathbf{t}) \le 1\right\}. \tag{4.35}$$

It was shown in [9] that for any p, M and Ψ satisfying the conditions of Theorem 4.2, the following equality holds:

$$\sup_{h\in\mathcal{U}_{p,M}(\mathbb{A})} \inf_{\gamma_\sigma\in\Gamma_\sigma} \inf\left\{\alpha>0: \int\limits_{\mathbb{A}\setminus\gamma_\sigma} M\Big(\tfrac{|\Psi(\mathbf{t})|y(\mathbf{t})}{\alpha}\Big) d\mu(\mathbf{t}) \le 1\right\} = \sup_{l\in(\sigma,a]} \frac{\left(\int_0^l \bar{\Psi}^{-p}(t)dt\right)^{-\frac{1}{p}}}{M^{-1}(\frac{1}{l-\sigma})}, \tag{4.36}$$

where M^{-1} is the inverse function of the function M and $\bar{\Psi}$ is the decreasing rearrangement of the function $|\Psi(\mathbf{t})|$, $\mathbf{t}\in\mathbb{A}$. The value of the exact upper bound in (4.34) is attained at some finite value $l=l^*$.

Combining relations (4.35) and (4.36), we obtain (4.34):

$$e_\sigma(\Psi U^p_\Phi)_{M,\Phi} \le \sup_{l\in(\sigma,a]} \frac{\left(\int_0^l \bar{\Psi}^{-p}(t)dt\right)^{-\frac{1}{p}}}{M^{-1}\Big(\frac{1}{l-\sigma}\Big)} = \frac{\left(\int_0^{l^*} \bar{\Psi}^{-p}(t)dt\right)^{-\frac{1}{p}}}{M^{-1}\Big(\frac{1}{l^*-\sigma}\Big)}.$$

Consider the function

$$y^*(\mathbf{t}) = \frac{\chi_{\mathbb{E}}(\mathbf{t})}{|\Psi(\mathbf{t})|}\bigg(\int_{\mathbb{E}} |\Psi(\mathbf{u})|^{-p} d\mu(\mathbf{u})\bigg)^{-\frac{1}{p}}, \qquad \mathbf{t}\in\mathbb{A},$$

where $\mathbb{E}$ is any subset of the set $\{\mathbf{t}\in\mathbb{A}:\ |\Psi(\mathbf{t})| \ge \bar{\Psi}(l^*-)\}$, which contains the set $\{\mathbf{t}\in\mathbb{A}: |\Psi(\mathbf{t})| > \bar{\Psi}(l^*-)\}$, $\mathrm{mes}_\mu\mathbb{E} = l^*$, and $\chi_{\mathbb{E}}$ is the characteristic function of the set $\mathbb{E}$.

Assume that the set $E(\Phi)$ of values of the operator Φ coincides with the space $L_M(\mathbb{A}, \mathrm{d}\mu)$. Then there exists an element $x\in\mathcal{S}_{M,\Phi}$ such that $\widehat{x}(\mathbf{t}) = y^*(\mathbf{t})$ almost everywhere on $\mathbb{A}$ and

$$\|x\|^p_{p,\Phi} = \|\widehat{x}\|^p_{L_p(\mathbb{A},\,\mathrm{d}\mu)} = \|y^*\|^p_{L_p(\mathbb{A},\,\mathrm{d}\mu)} = \int_{\mathbb{E}} |\Psi(\mathbf{t})|^{-p}\, \mathrm{d}\mu(\mathbf{t})\bigg(\int_{\mathbb{E}} |\Psi(\mathbf{u})|^{-p}\, \mathrm{d}\mu(\mathbf{u})\bigg)^{-1} = 1,$$

therefore, $x\in U^p_\Phi$. Furthermore, there exists an element $x_\Psi\in\mathcal{S}_{M,\Phi}$ such that almost everywhere on $\mathbb{A}$

$$\widehat{x}_\Psi(\mathbf{t}) = \Psi(\mathbf{t})\widehat{x}(\mathbf{t}) = \Psi(\mathbf{t})y^*(\mathbf{t}).$$

Since $x\in U^p_\Phi$, we have $\widehat{x}_\Psi \in \Psi U^p_{M,\Phi}$ and the following relation holds:

$$\begin{aligned} e_\sigma(x)_{M,\Phi} &= \inf_{\gamma_\sigma\in\Gamma_\sigma} \|\Phi(x_\Psi - U_{\gamma_\sigma}(x_\Psi))\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)} \\ &= \inf_{\gamma_\sigma\in\Gamma_\sigma} \|\widehat{x}_\Psi - \chi_{\gamma_\sigma}\widehat{x}_\Psi\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)} \\ &= \inf_{\gamma_\sigma\in\Gamma_\sigma} \|\Psi y^* - \chi_{\gamma_\sigma}\Psi y^*\|_{L_M(\mathbb{A},\,\mathrm{d}\mu)} \\ &= \inf_{\gamma_\sigma\in\Gamma_\sigma} \inf\left\{\alpha>0: \int_{\mathbb{A}\setminus\gamma_\sigma} M\Big(\tfrac{|\Psi(\mathbf{t})|y^*(\mathbf{t})}{\alpha}\Big)\mathrm{d}\mu(\mathbf{t}) \le 1\right\} \\ &= \inf\left\{\alpha>0:\ (l^*-\sigma)M\bigg(\bigg(\int_{\mathbb{E}} |\Psi(\mathbf{u})|^{-p}\,\mathrm{d}\mu(\mathbf{u})\bigg)^{-\frac{1}{p}}/\alpha\bigg) \le 1\right\}. \end{aligned}$$

For any number $h\ge 0$, the μ-measure of the set $\{\mathbf{t}\in\mathbb{E}:\ |\Psi(\mathbf{t})|\ge h\}$ is equal to the Lebesgue measure of the set $\{t\in(0,s^*):\ \bar{\Psi}(t)\ge h\}$. Therefore, from the definition of the decreasing rearrangement of a function, it follows that

$$\int_{\mathbb{E}} |\Psi(\mathbf{u})|^{-p}\, \mathrm{d}\mu(\mathbf{u}) = \int_0^{s^*} \bar{\Psi}^{\,-p}(t)\, \mathrm{d}t$$

and

$$e_\sigma(x)_{M,\Phi} = \inf\left\{\alpha > 0 : \ M\left(\left(\int_0^{s^*} \bar{\varphi}^{-p}(t)dt\right)^{-\frac{1}{p}}/\alpha\right) \le \tfrac{1}{s^*-\sigma}\right\} = \frac{\left(\int_0^{l^*} \bar{\Psi}^{-p}(t)dt\right)^{-\frac{1}{p}}}{M^{-1}\left(\frac{1}{l^*-\sigma}\right)}.$$

Thus, in the given case, the relation (4.34) is actually an equality.

□

Note that in the spaces $\mathcal{S}^p_\Phi$, statements similar to Theorem 4.2 were obtained in [12, 16, 8]. In particular, if $M(u) = u^q$, $q = p$, the statement of Theorem 4.2 coincides with the statement of Theorem 2 in [12]. In the case when $0 < p, q < \infty$, the results similar to Theorem 4.2 were obtained in [16]. Estimates similar to (4.36) in the spaces $L_p(\mathbb{A}, \mathrm{d}\mu)$ were proven in [17]. In the spaces $\mathcal{S}^p_\varphi$ and $\mathcal{S}^p(\mathbb{T}^d)$, statements similar to Theorem 4.1 and 4.2 were obtained in [11, 13] and [14, Chap. 11], and in the Orlicz sequence spaces l_M, such statements were proven in [10].

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